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The elastodynamic Liénard–Wiechert potentials and elastic fields of non-uniformly moving point and line forces

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ABSTRACT

The purpose of this paper is to investigate the fundamental problem of the non-uniform subsonic motion of a point force and of line forces in an unbounded, homogeneous, isotropic medium in analogy to the electromagnetic Liénard–Wiechert potentials. The exact closed-form solutions of the displacement field and of the elastic fields produced by the point force and the line forces are calculated. The displacement fields can be identified with the elastodynamic Liénard–Wiechert tensor potentials. For a non-uniformly moving point force, we decompose the elastic fields into a radiation part and a non-radiation part. We show that the solution of a non-uniformly moving point force is the generalization of the Stokes solution towards the non-uniform motion. For line forces, the mathematical solutions are given in the form of time integrals and, therefore, their motion depends on the history.

1. Introduction

An important item in elastodynamics is concerned with the radiation and the waves produced by the non-uniform motion of body forces. This is a fascinating and interdisciplinary research topic. The radiation problem has attracted the interest of researchers from different fields such as applied mathematics, material science, continuum mechanics, and seismology (see, e.g., [1–5]). A fundamental question is: what is the elastic radiation caused by non-uniformly moving point forces?

In elastostatics, the so-called Kelvin problem concerns the displacement field and the elastic fields produced by a static point force. In elastodynamics the displacement field generated by a time-dependent concentrated point load was first presented by Stokes [6] (see, e.g., [1,4,7]). In the Stokes problem, the body force is considered as a concentrated load of time-dependent magnitude. The Stokes solution can be considered as the first mathematical model of an earthquake [8]. Concentrated line forces with time-dependent magnitude were studied by Achenbach [1] and de Hoop [9]. The wave-motion caused by a line force moving non-uniformly in a fixed direction was considered by Freund [10]. A non-uniformly moving line force in an anisotropic elastic solid was studied by Wu [11].

The radiation problem of point forces is three dimensional, so Huygens' principle prevails. Using the Helmholtz decomposition, the so-called retarded potentials were given for the waves produced by body forces in elastodynamics (see, e.g., [1,2]). A more general expression for the retarded potential in elastodynamics was given by Hudson [3]. Elastodynamic fields propagate with finite velocities. There always is a time delay before a change in elastodynamic conditions initiated at a point of space can produce an effect at any other point of space. This time delay is called elastodynamic retardation.

In electrodynamics, radiation is caused by the non-uniform motion of an electric point charge. The electric and magnetic potentials of such a non-uniformly moving point charge are called the Liénard–Wiechert potentials. The corresponding

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electric and magnetic field strengths consist of velocity-dependent fields and acceleration-dependent fields. The latter are the fields of radiation. This is a standard topic in electromagnetic field theory and is covered in a lot of books on electrodynamics (e.g. [12,13]). It is quite surprising that nothing has been investigated in this direction in the elastodynamics of moving point forces. No solution of a non-uniformly moving point force analogous to the Liénard–Wiechert potential can be found in standard books on elastic waves (e.g. [1,2,14,15,3–5]).

The purpose of the present paper is to investigate the fundamental problem of the non-uniform motion of a point force as well as line forces in an unbounded, homogeneous, isotropic medium in analogy to the electromagnetic Liénard–Wiechert potentials. We consider subsonic motion. The paper is organized as follows. In Section 2, we present the framework of elastodynamics and we formulate the equation of motion. In Section 3, using the three-dimensional elastodynamic Green tensor, we calculate the elastodynamic Liénard–Wiechert potential of a point force. In Section 4, using the Liénard–Wiechert potential of a point force, we determine the elastic distortion and the velocity fields (particle velocity) of the medium caused by the non-uniformly moving point force. In addition, we specify the radiation fields proportional to the acceleration of the point force. The limit to the Stokes solution is performed in Section 5. The static limit of the displacement field and of the elastic fields of a non-uniformly moving point force is given in Section 6. In Section 7, using two-dimensional Green tensors, we give the general solution of two-dimensional non-uniformly moving line forces. We close the paper with conclusions in Section 8.

2. The elastodynamic equation of motion

In elastodynamics [7], the force balance law reads[1]

$$\dot{p}_i - \partial_j \sigma_{ij} = F_i, \tag{1}$$

where $\boldsymbol{p}$, $\boldsymbol{\sigma}$, and $\boldsymbol{F}$ are the linear momentum vector, the force stress tensor, and the body force vector. In the theory of linear elasticity, the momentum vector $\boldsymbol{p}$ and the stress tensor $\boldsymbol{\sigma}$ can be expressed in terms of physical state quantities, namely, the velocity vector (particle velocity) $\boldsymbol{v} = \dot{\boldsymbol{u}}$ and the elastic distortion tensor $\boldsymbol{\beta} = (\mathrm{grad}\,\boldsymbol{u})^T$ of the medium, which can be derived from a displacement vector $\boldsymbol{u}$ by means of the following constitutive relations:

$$p_i = \rho\, v_i = \rho\, \dot{u}_i, \tag{2}$$

$$\sigma_{ij} = C_{ijkl}\, \beta_{kl} = C_{ijkl}\, \partial_l u_k, \tag{3}$$

where ρ denotes the mass density and C_{ijkl} is the tensor of elastic moduli. The tensor C_{ijkl} possesses the following symmetry properties:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{4}$$

If we substitute the constitutive relations (2) and (3) in Eq. (1), we obtain the force balance law expressed in terms of the displacement vector $\boldsymbol{u}$:

$$\rho\, \ddot{u}_i - C_{ijkl}\partial_j\partial_l u_k = F_i. \tag{5}$$

The solution of Eq. (5) can be represented as a convolution integral in space and time. In an unbounded medium, and under the assumption of zero initial conditions, which means that $\boldsymbol{u}(\boldsymbol{r}, t_0)$ and $\dot{\boldsymbol{u}}(\boldsymbol{r}, t_0)$ are zero for $t_0 \to -\infty$, the solution of $\boldsymbol{u}$ reads

$$u_i(\boldsymbol{r}, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} G_{ij}(\boldsymbol{r} - \boldsymbol{r}', t - t')\, F_j(\boldsymbol{r}', t')\, d\boldsymbol{r}'\, dt'. \tag{6}$$

Here, G_{ij} is the elastodynamic Green tensor of the anisotropic Navier equation defined by

$$\left[\delta_{ik}\, \rho\, \partial_{tt} - C_{ijkl}\partial_j\partial_l\right] G_{km} = \delta_{im}\, \delta(t)\delta(\boldsymbol{r}), \tag{7}$$

where $\delta(\cdot)$ denotes the Dirac delta function and δ_{ij} is the Kronecker delta. The tensor of elastic moduli for isotropic materials is given by

$$C_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\, \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{8}$$

where λ and μ are the Lamé constants. Substituting Eq. (8) in Eqs. (5) and (7), we obtain respectively the isotropic Navier equations for the displacement vector

$$\left[\delta_{ij}\, \rho\, \partial_{tt} - \delta_{ij}\, \mu\, \Delta - (\lambda + \mu)\, \partial_i\partial_j\right] u_j = F_i, \tag{9}$$

and for the elastodynamic Green tensor

$$\left[\delta_{ij}\, \rho\, \partial_{tt} - \delta_{ij}\, \mu\, \Delta - (\lambda + \mu)\, \partial_i\partial_j\right] G_{jm} = \delta_{im}\, \delta(t)\delta(\boldsymbol{r}), \tag{10}$$

where Δ denotes the Laplacian.

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[1] Spatial differentiation is denoted by $\partial_j \equiv \partial/\partial x_j$, and for differentiation with respect to time t we use the notation $\dot{p}_i \equiv \partial_t p_i$.