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On an inverse source problem connected with photo-acoustic and thermo-acoustic tomographies

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ABSTRACT

An inverse source problem which aims to determine the source density $p_0(\mathbf{x})$ taking place in the wave equation $\Delta p(\mathbf{x}, t) - (1/c^2)\partial^2 p(\mathbf{x}, t)/\partial t^2 = -p_0(\mathbf{x})\delta'(t)$ is considered. One assumes that $p_0(\mathbf{x})$ is a function of bounded support while $p(\mathbf{x}, t)$ can be measured on the boundary S of a *convex* domain D during a certain *finite* time interval [0,T]. An explicit expression of the solution is given in terms of the surface integral of the data on S. Two illustrative examples show the applicability as well as the effectiveness of the method. In one of these examples S consists of a spheroid while in the other it consists of a half of the spheroid and a disc. The problem is motivated by photo-acoustic and thermo-acoustic applications.

1. Introduction

Consider the wave equation

$$\Delta p(\mathbf{x}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}p(\mathbf{x}, t) = -p_0(\mathbf{x})\delta'(t) \tag{1}$$

satisfied by the wave function $p(\mathbf{x}, t) \in C^2(R^3 \times R)$ under the *radiation conditions* to be stated later on for $|\mathbf{x}| \to \infty$. Here $\mathbf{x} = (x_1, x_2, x_3) \in R^3$ and $t \in R$ stand for any point in three-dimensional space and time, respectively, while c is a positive constant (the propagation velocity of the wave). The function appearing on the right hand side, i.e. $p_0(\mathbf{x})\delta'(t)$, is the time-derivative of the source density, which excites the wave. In the problem to be considered here, $p_0(\mathbf{x}) \in L^1(V_0)$ is assumed to be a function of *bounded support* with support $V_0 \subset R^3$ while $\delta(t)$ is the classical Dirac distribution. In a direct problem the source density $p_0(\mathbf{x})$ is assumed to be known and one tries to find the wave function $p(\mathbf{x}, t)$ for all $\mathbf{x} \in R^3$ and $t \in R$. But here, in the *inverse source problem* that we will consider, the situation is converse: we will assume that the wave function $p(\mathbf{x}, t)$ is known (by measurements) during certain time interval $[0, T]$ on a surface S, which involves the region V_0 inside, and try to determine the source density $p_0(\mathbf{x})$. The result will also reveal the support V_0 where the source is located.

The inverse problem stated above was motivated by the so-called *photo-acoustic* and *thermo-acoustic tomographies* which have important applications in medicine. So there is extensive research and many publications on the subject (see for ex. [1–11] and references cited there). In tomographic applications the above-mentioned support V_0 consists of an anomalous (for example cancerous) domain in a soft biological region. Then the surface S is the boundary of a soft biological region. Although the latter may have various shapes, to the best of our knowledge almost all previous theoretical

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investigations are devoted to very simple but unsuitable shapes. Among them we can mention, for example, the regions bounded by two parallel *infinitely large* planes or *infinitely long* circular cylinders or *spheres,* which require quite different reconstruction formulas (see for ex. [4–10]). The aim of the present paper is to consider a more general and plausible case where the shape of the region bounded by S is arbitrary but *convex,* and give an *exact* formula which needs the surface integration of the data collected on S during a finite time interval. Our result is an extension of those due to Xu and Wang [9,10]. In [9] the authors consider three particular cases separately: (i) a region bounded by two parallel planes, (ii) a region bounded by an infinitely long circular cylinder, and (iii) a region bounded by a sphere. To carry out computations, they use expansions (in the form of infinite series or integrals) involving exponential, Bessel and Legendre functions, respectively. In the present work we give a unified most general and rather simple expression in terms of a surface integral.

In what follows we will assume that the *reflection* of the field $p(\mathbf{x}, t)$(=*the pressure*) on S (=the boundary of the biological medium) is *negligibly small,* which permits us to formulate the problem in the whole of the space. This assumption may be tolerable in biological applications where the tissue is soft and the velocity of propagation of the pressure wave is moderate. In the case where this assumption is not met, one has to reconsider the problem with *transmission type* boundary conditions, which, to the best of our knowledge, constitutes an open problem although in the open literature there are very few available papers devoted to this type of investigations (see for ex. [12]).

Since the solution of the inverse problem is always based on the solution to the direct problem, in order to make the paper self-contained, in what follows we will first consider the direct problem and derive an explicit expression for its solution (see Section 2). Then in Section 3 we will obtain the exact results pertinent to the inverse problem. In Section 4 we will propose an algorithm which may be appropriate to numerical implementation. Two illustrative examples which show the applicability as well as the effectiveness of the method are given in Section 5. Finally, in Section 6 one recapitulates the main results obtained here and one indicates some open problems which merit of investigation.

In our discussion the Fourier integral transform as well as the distribution concept will play important roles. To denote the Fourier transform of a function $f(t) \in L^1(-\infty, \infty)$, we will use a hat on the letter f, namely:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt, \quad \omega \in (-\infty, \infty). \tag{2a}$$

It is known that at every point t where $f(t)$ is continuous, (2a) is inverted uniquely as follows:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{-i\omega t} d\omega, \quad t \in (-\infty, \infty). \tag{2b}$$

As to the distributions, a widely used one is the Dirac distribution $\delta(t)$ and some of its generalizations, namely:

$$t\delta'(t) = -\delta(t), \tag{3a}$$

$$\widehat{\delta} = 1, \tag{3b}$$

$$\widehat{\delta'} = -i\omega \tag{3c}$$

and

$$\begin{aligned} \Delta\left(\frac{e^{\pm ikR}}{R}\right) + k^2\left(\frac{e^{\pm ikR}}{R}\right) &= -4\pi\,\delta(\mathbf{x}-\mathbf{y}) \\ &= -4\pi\,\delta(x_1-y_1)\delta(x_2-y_2)\delta(x_3-y_3). \end{aligned} \tag{4a}$$

Here R stands for the distance between the points $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$:

$$R = |\mathbf{x}-\mathbf{y}| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}. \tag{4b}$$

From (3c) we also write

$$\int_{-\infty}^{\infty} e^{\pm ikR}\omega d\omega = -(\pm 2\pi i)\delta'\left(\frac{R}{c}\right), \qquad \int_{-\infty}^{\infty} \left[e^{-ikR} - e^{ikR}\right]\omega d\omega = 4\pi i\delta'\left(\frac{R}{c}\right), \tag{4c}$$

where $k = \omega/c$ while R is given by (4b).

2. Solution of the direct problem

Let us take the Fourier transform of both sides of (1) with respect to t. Owing to the definition (2a) and known relations (3b), we get

$$\Delta\, \widehat{p}(\mathbf{x}, \omega) + k^2\, \widehat{p}(\mathbf{x}, \omega) = i\omega p_0(\mathbf{x}), \quad \mathbf{x} \in R^3. \tag{5}$$