LINEAR AND NONLINEAR DISSIPATIVE DYNAMICS

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In this paper we introduce and study new dissipative dynamics for large interacting systems.

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1. Introduction

The theory of large dissipative systems has a long and growing mathematical history. Some of the classical literature one could find e.g. in [24] and [37]; see also references therein. In this paper we focus on dissipative dynamics with noncompact configuration space and their counterparts in noncommutative algebras.

A construction of Markov semigroups on the space of continuous functions with an infinite-dimensional underlying space well suited to study strong ergodicity problems can be found in [51] in case of fully elliptic generators. More recently it was extended to subelliptic situation in [16, 31] and Lévy type generators [35]. An interesting approach via stochastic differential equations one can find in [15] and some recent extension to subelliptic generators in [50] (see also [3, 4] and references therein). Another approach via Dirichlet forms theory which is well adapted to L_2 theory, can be found e.g. in [1, 45] and reference therein.

For symmetric semigroups, after a recent progress in proving the log-Sobolev inequality for infinite-dimensional Hörmander type generators $\mathcal{L}$ symmetric in $L_2(\mu)$ defined with a suitable nonproduct measure μ ([25–28, 32, 43]), one can expect an extension of the established strategy [51] for proving strong pointwise ergodicity for the corresponding Markov semigroups $P_t \equiv e^{t\mathcal{L}}$ (respectively in the uniform norm in case of the compact spaces as in [24] and references therein). One could obtain more results in this direction, including configuration spaces given by infinite products of general noncompact nilpotent Lie groups other than Heisenberg type groups, by conquering a (finite-dimensional) problem of sub-Laplacian bounds (of the corresponding control distance) which for a moment remains still very hard.

The ergodicity theory in case when an invariant measure is not given in advance, in noncompact subelliptic setup is an interesting and challenging problem which was initially studied in [16] and was extended in new directions in [31] developing further strategy based on generalised gradient bounds. We remark that in fully elliptic case a strategy based on classical Bakry–Emery arguments involving restricted class of interactions can be achieved. In case of the stochastic strategy of [15], the convexity assumption enters via dissipativity condition in a suitable Hilbert space and does not improve the former one as far as ergodicity is concerned; (on the other hand it allows to study a number of stochastically natural models). In subelliptic setup involving subgradient this strategy faces serious obstacles, see e.g. comments in [6].

In noncommutative setup the development of mathematical description of infinite dissipative systems is much less developed. Some description of infinite-dimensional dissipative dynamics of jump type which are not symmetric with respect to a given Gibbs state as well as some results on theirs ergodicity can be found in [54]; see also references therein and [14, 23, 38] on constructions associated to classical Gibbs states (where interaction potential is classical). In [40] a construction and ergodicity results were provided for an interesting class where generator of jumps part corresponds to a classical potential, but additionally the generator contains a conservative part corresponding to a different possibly nonclassical potential. In general, for an infinite-dimensional system still no construction of jump type dynamics exists which would be symmetric for a Gibbs state associated to a generic nonclassical potential. Some interesting general constructions, based on application of Dirichlet form theory [13], are provided in [14, 44] (see also references therein).

A study of diffusion type dynamics providing a construction and ergodicity results were given in [34], including generators associated to a family of noncommuting fields, but not a priori symmetric with respect to an $\mathbb{L}_2$ scalar product associated to a given state.

Another recent examples of dissipative dynamics for infinite boson systems can be found in [7, 41].

One of the important techniques developed to study ergodicity of dissipative dynamics of infinite classical interacting systems is based on use of hypercontractivity property or its infinitesimal form encoded in log–Sobolev inequality ([24] and references therein). A noncommutative basis for such theory was introduced in [42]. Since then, in noncommutative setup some progress was achieved in studying certain directions ([2, 9, 11, 12]) with interesting new results emerging in connection to quantum information theory ([29, 30]). Still many important technical aspects necessary to effective implementation of the theory remain elusive in noncommutative world. (This includes e.g. the product and perturbation properties of log–Sobolev inequality.)

In Sections 2 and 3, we study finite- and infinite-dimensional systems for which we construct dissipative dynamics described by Dunkl type generators and provide certain basic ergodicity results. In Section 4 we give an example of such dissipative dynamics in noncommutative setup. In Section 5 we discuss some nonlinear classical dissipative dynamics and theirs noncommutative counterparts. In Appendix we provide some discussion of monotone convergence in noncommutative $\mathbb{L}_p$ spaces.