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OPERATOR REFLECTION POSITIVITY INEQUALITIES AND THEIR APPLICATIONS TO INTERACTING QUANTUM ROTORS

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In the Reflection Positivity theory and its application to statistical mechanical systems, certain matrix inequalities play a central role. The Dyson–Lieb–Simon [1] and Kennedy–Lieb–Shastry [2] inequalities constitute prominent examples. In this paper we extend the KLS inequality to the case where matrices are replaced by certain operators. As an application, we prove the occurrence of the long-range order in the ground state of two-dimensional quantum rotors.

Keywords: statistical mechanics; phase transitions; operator inequalities; reflection positivity.

1. Introduction

The notion of *Reflection Positivity* has appeared in quantum field theory in the seventies of the last century [3]. Few years later, it has been applied to investigation of phase transitions in both classical [4] and quantum [1] lattice spin systems. The reflection positivity turned out to be a very useful tool, giving the first rigorous proofs of existence of phase transitions in systems with continuous symmetry group.

The cornerstone of reflection positivity for quantum spin systems is the matrix inequality due to Dyson, Lieb and Simon (Lemma 4.1 in [1]). Using their inequality, the authors proved the existence of orderings in the XY as well as Heisenberg models in $d \geq 3$ and for sufficiently small temperature. Later on, this method has been extended to certain class of *infinite-dimensional* operators. This way, the existence of long-range order has been proved for $d \geq 3$ in the system of quantum interacting rotors [5].

Another direction of development of the reflection positivity techniques was an examination of *ground states* of quantum spin systems and orderings therein. It turned out that one can take certain zero-temperature limit in the framework of the DLS method. This way, the appearance of long-range order has been proved in XY and Heisenberg models in $d = 2$ [6–8]. Later on, it turned out that such a proof can be done directly in the ground state, with the use of another matrix inequality, due to Kennedy, Lieb and Shastry (KLS) [2]. This inequality was further generalized by Schupp [9].

It would be tempting to extend this inequality to infinite-dimensional version, i.e. for certain class of operators. However, to our best knowledge, the operator version of the KLS and Schupp (KLSS) inequalities, suitable for applications to ground states of quantum interacting rotors, has *not* been developed.

This opportunity inspired us to attempts to prove an operator analog of the KLSS inequalities. It turned out to be possible, and this is one of two main results of our paper: *extension of the KLSS matrix inequalities to certain class of infinite-dimensional operators*. The second group of results which seems to be new are some applications.

The outline of the paper is as follows. In Section 2 we formulate the operator version of the KLSS inequalities. The application of these operator inequalities is described in Section 3; it is the proof of the ordering in ground state of $d \geq 2$ rotors (alternative proof to that presented in [10]). Section 4 contains summary, conclusions and description of some open problems.

2. KLS inequality and its extension for operators

2.1. Kennedy, Lieb, Shastry and Schupp matrix inequalities

For convenience of the reader, and to show the idea of a proof without operator-theoretic details, we present firstly the matrix KLSS inequalities.

THEOREM 2.1. [2] *Let c, A, B be $n \times n$ complex matrices, $|c| := \sqrt{c^*c}$ and $|c^*| := \sqrt{cc^*}$ the moduli of c and c^*, respectively. Then*

$$|\mathrm{Tr}\, c^* B c A^*| \leq \frac{1}{2}\left[\mathrm{Tr}\,(|c|A|c|A^*) + \mathrm{Tr}\,(|c^*|B|c^*|B^*)\right]. \tag{1}$$

Sketch of the proof: At first let us note that by the polar decomposition theorem c is of the form $c = u|c|$, where u is a partial isometry. Since $u^*u|c| = |c|$ and $u|c|u^*$ is a positive matrix, the polar decomposition of c^* is of the form

$$c^* = |c|u^* = u^*u|c|u^* = u^*|c^*|. \tag{2}$$

Taking adjoint we get $c = u|c| = |c^*|u$. Therefore

$$u\sqrt{|c|} = \sqrt{|c^*|}u \tag{3}$$

according to functional calculus of positive hermitian matrices and

$$c = \sqrt{|c^*|}u\sqrt{|c|}. \tag{4}$$