GENERALIZED HOPF FIBRATION AND GEOMETRIC SO(3) REDUCTION OF THE 4DOF HARMONIC OSCILLATOR

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It is shown that the generalized Hopf map $\mathbb{H}\times\mathbb{H}\to\mathbb{H}\times\mathbb{R}\times\mathbb{R}$ in quaternion formulation can be interpreted as an SO(3) orbit map for a symplectic SO(3) action. As a consequence the generalized Hopf fibration $\mathbb{S}^7\to\mathbb{S}^4$ appears in the SO(3) geometric symplectic reduction of the 4DOF isotropic harmonic oscillator. Furthermore it is shown how the Hopf fibration and associated twistor fibration play a role in the geometry of the Kepler problem and the rigid body problem.

Keywords: Hopf map, Hopf fibration, symplectic reduction, harmonic oscillator, Kepler problem, rigid body.

1. Introduction

It is well known that one may reduce the phase space of the 2DOF harmonic oscillator by using the symmetry given by the $\mathbb{S}^1$-action generated by the flow of the Hamiltonian. Restricting the reduction mapping to the 3-spheres given by the energy surfaces, the orbit mapping becomes a reduction of the energy surface $\mathbb{S}^3$ to the reduced phase space $\mathbb{S}^2$, which mapping is exactly the Hopf fibration $\mathbb{S}^3\to\mathbb{S}^2$, see [4]. In Section 2 we will make this precise starting from the classical description of the Hopf fibration in complex coordinates. In the following sections we generalize these ideas to the 4DOF isotropic harmonic oscillator. For this purpose we will introduce the generalized Hopf map in terms of quaternions in Section 3. In Section 4 we will show that, after choosing an appropriate real representation, this Hopf map can be considered as an orbit map for an appropriately chosen symplectic SO(3) action, and that restriction to an energy surface for the harmonic oscillator, gives the generalized Hopf fibration $\mathbb{S}^7\to\mathbb{S}^4$. In Section 5 we consider the twistor fibration as it appears in connection to the reduction of the harmonic oscillator. In Section 6 we discuss how the geometry of the Hopf map is related to the geometry of the phase space of the Kepler system and the rigid body system. The SO(3) reduction given by the generalized Hopf fibration can be considered as a reduction map for the rigid body problem and we show how the geometry of the Kepler problem is connected to this Hopf fibration reduction through the twistor fibration.

In the final section we discuss some future work where these ideas are used for studying perturbed Kepler and rigid body systems in the context of the full gravitational 2-body system.

2. Hopf fibration in 2DOF

In this part we briefly recall the well-known Hopf fibration and Hopf map in the complex plane, which can be shown to be an orbit map and a reduction map for a 2DOF harmonic oscillator. With this purpose, we consider the complex plane $(\mathbb{C}^2, \langle\,\rangle)$, where $\langle z, w\rangle = \bar{z}w = \bar{z}_1 w_1 + \bar{z}_2 w_2$ is the usual hermitian inner product. Let us also consider the equivalence relation on the 3-sphere $\mathbb{S}^3 = \{z \in \mathbb{C}^2 \mid \langle z, z\rangle = 1\}$ given by $z \sim \omega$ iff $\omega = \lambda z$, $\lambda \in \mathbb{S}^1$. Then, we get that

$$\mathbb{S}^3/\!\sim\; \cong P(\mathbb{C}^2) = \mathbb{CP}^1.$$

In what follows we denote by $\mathcal{A}$ the Hopf map and by $\mathcal{F}_{\mathbb{C}}$ the complex Hopf fibration, which is built in a constructive process portrayed in Fig. 1. The Hopf map is explicitly given and is obtained as the composition of $P_\sim$ and Π.

Note that Π is based on the classic stereographic projection from $\mathbb{S}^2(\frac{r}{2}) - N$ onto the real plane, but in this case the north pole is covered by the infinite point $[(1, 0)]$.

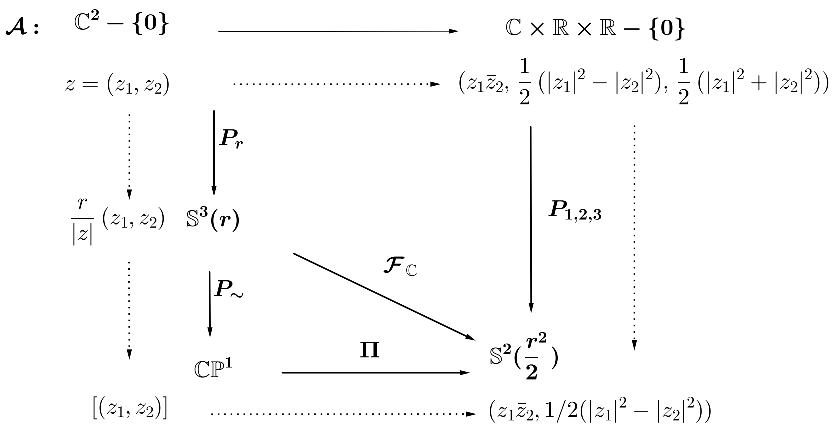

Fig. 1. Hopf map and classic Hopf fibration diagram. $P_\sim$ identifies each point in the sphere $\mathbb{S}^3(r)$ to its corresponding class of equivalence on $\mathbb{CP}^1$, Π is the stereographic projection, P_r is the map from $\mathbb{C}^2 - \{0\}$ to $\mathbb{S}^3(r)$ matching each semi-ray through the origin with the corresponding element of module equal to r in $\mathbb{S}^3(r)$ and $P_{1,2,3}$ is the projection of $\mathbb{C} \times \mathbb{R} \times \mathbb{R} - \{0\}$ into $\mathbb{C} \times \mathbb{R}$.