MAXIMUM-ENTROPY INFERENCE AND INVERSE CONTINUITY OF THE NUMERICAL RANGE

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We study the continuity of the maximum-entropy inference map for two observables in finite dimensions. We prove that the continuity is equivalent to the strong continuity of the set-valued inverse numerical range map. This gives a continuity condition in terms of analytic eigenvalue functions which implies that discontinuities are very rare. It shows also that the continuity of the *MaxEnt* inference method is independent of the prior state.

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1. Introduction

The maximum-entropy principle, going back to Boltzmann, is one of the standard techniques in quantum mechanical inference problems [2, 15, 16, 39, 48] and state reconstruction [6, 40]. Here we consider a tuple of quantum observables, represented by hermitian matrices in the algebra M_d of complex $d \times d$-matrices, $d \in \mathbb{N}$. Their expected values provide partial information about the state of a quantum system. Several states may have the same tuple of expected values, so an inference rule is needed to select a unique state from its expected values. The maximum-entropy inference map selects the state with maximal von Neumann entropy.

The maximum-entropy inference map is continuous if the observables commute [45], an example being the inference of probability distributions from expected values of random variables. Surprisingly, discontinuity points exist in the noncommutative case on the boundary of the set of expected values [47]. They have been discussed as signatures of quantum phase transitions [8] and are passed [32, 46] to a correlation quantity, called irreducible correlation [23, 49], which is useful, beside the topological entanglement entropy, to characterize topological order [18, 25].

The methods to analyze discontinuities have included information topology [44], convex geometry [32, 45], and, for two observables, numerical range techniques [32]. Here, we focus on the case of two observables which we encode into a single matrix

$A \in M_d$ as its *real part* $\Re(A) := \frac{1}{2}(A + A^*)$ and *imaginary part* $\Im(A) := \frac{1}{2i}(A - A^*)$, a notation which we will meet again in Section 6. The set of *density matrices* in M_d is denoted by

$$\mathcal{M}_d := \{\rho \in M_d \mid \rho \succeq 0, \ \mathrm{tr}(\rho) = 1\}.$$

This set is also called *state space* [1], $a \succeq 0$ means that the matrix $a \in M_d$ is positive semi-definite. The state space is a convex body, that is a compact convex subset in a Euclidean space. The inner product $\langle a, b\rangle := \mathrm{tr}(a^* b)$, $a, b \in M_d$, and the norm $\|a\|_2 := \sqrt{\langle a, a\rangle}$ shall be used.

Elements of $\mathcal{M}_d$ represent *states* of a quantum system, see for example [4], Sections 5.1 and 5.2. The real number $\mathrm{tr}(\rho a) = \langle \rho, a\rangle$, for an observable $a \in M_d$ and for $\rho \in \mathcal{M}_d$, is interpreted as the expected value of a when the system is in the state ρ. The map $\rho \mapsto \langle \rho, A\rangle$ will be used with various restrictions. Since the simplest notation for a restricted map is to use a single-letter function symbol, we define the *expected value functional*

$$\mathbb{E}_A : \quad \{b \in M_d \mid b^* = b\} \to \mathbb{C}, \qquad a \mapsto \langle a, A\rangle,$$

on the Euclidean space of hermitian matrices. The map $\mathbb{E}_A$ sends a state $\rho \in \mathcal{M}_d$ to the pair $\mathbb{E}_A(\rho) = (\langle \rho, \Re(A)\rangle, \langle \rho, \Im(A)\rangle)$ of expected values of the observables $\Re(A)$ and $\Im(A)$, in the identification of the range $\mathbb{C}$ with $\mathbb{R}^2$.

The domain of the maximum-entropy inference map is the convex body

$$L_A := \{\mathbb{E}_A(\rho) \mid \rho \in \mathcal{M}_d\} \subset \mathbb{R}^2,$$

comprising the expected value pairs of $\Re(A)$ and $\Im(A)$. We call L_A *convex support* [32, 45, 47] by its name in probability theory [3]. The *von Neumann entropy* of a state $\rho \in \mathcal{M}_d$ is $S(\rho) = -\mathrm{tr}(\rho \cdot \log \rho)$ and the *maximum-entropy inference* is the map

$$\rho_A^* : \quad L_A \to \mathcal{M}_d, \quad \alpha \mapsto \mathrm{argmax}\{S(\rho) \mid \rho \in \mathcal{M}_d, \mathbb{E}_A(\rho) = \alpha\}.$$

See [15, 16] for more information about ρ_A^*. Our analysis will be based on [45, Theorem 4.9], which affirms that for all $\alpha \in L_A$

$$\rho_A^* \text{ is continuous at } \alpha \text{ if, and only if, } \mathbb{E}_A|_{\mathcal{M}_d} \text{ is open at } \rho_A^*(\alpha). \tag{1.1}$$

Thereby, a function between topological spaces is *open at* a point in the domain, if the image of every neighborhood of that point is a neighborhood of the image point. Clearly, every linear map is open in finite dimensions but it may fail to be open when restricted.

Exact bounds on the number of discontinuity points of ρ_A^* are known for $d \leq 5$, see Sections 6 and 7 of [32]. The bounds have been derived from pre-image results [20] of the following map f_A. The aim of this article is to go beyond these pre-image results and to establish a direct link to a continuity problem in operator theory [10, 21, 24]. Denoting by $S\mathbb{C}^d$ the unit sphere of $\mathbb{C}^d$, the *numerical range map* of a matrix $A \in M_d$ is defined by

$$f_A : \quad S\mathbb{C}^d \to \mathbb{C}, \qquad x \mapsto \langle x, Ax\rangle.$$