NOT-QUITE-HAMILTONIAN REDUCTION

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The not-quite-Hamiltonian theory of singular reduction and reconstruction is described. This includes the notions of both regular and collective Hamiltonian reduction and reconstruction.

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1. Introduction

Ever since the beginning of analytical mechanics, there has been an effort to understand how to reduce the equations of motion given the presence of constraints of one form or another. For example, the motion of a particle that is described by Lagrangian or Hamiltonian formalism that is constrained to move on a submanifold of configuration space will, by employing D'Alembert's principle, give a reduced system of equations of the same form. A much more sophisticated, and perhaps the most striking early example of such considerations was Jacobi's elimination of the node in the three body problem. Such examples showed that there was a relation between symmetry and conservation laws, and these were explained for variational problems by Noether in her important work [25]. Somewhat dual to this, because of the Hamilton–Jacobi theory, there was an evolving understanding of the nature of symmetry and conservation laws on the Hamiltonian side, especially in understanding non-abelian symmetry groups and the reduction of Hamilton's equations. The first serious counterpart to Noether's theorem on the Hamiltonian side was the paper of Meyer [23].[1] In this paper, Meyer showed that the free and proper Hamiltonian action of a connected Lie group on a symplectic manifold led to a reduced symplectic manifold and the reduced dynamics was Hamiltonian. The importance of this theorem is the realization that the structure of the equations of motion has been reproduced under reduction by symmetry. This is a recurrent theme in further work. This work was followed by a torrent of papers on reduction, all with somewhat different emphases. For example, some, such as Marsden and Weinstein [22], stressed the

[1] Such a judgement call is always to some extent a question of taste. The reader may have some sympathy for our point of view after rereading the earlier works of Arnol'd [2] and Smale [29].

role of the momentum map, while others, such as Churchill, Kummer and Rod [9], looked at the relations of symmetry to averaging. During the 1980s and 1990s there was a growing awareness of the need to include singularity and the desire to discuss dynamics on the reduced space. A key observation in this time was that the dynamics on the reduced space could be described by the Poisson bracket on the invariant functions. Some of the notable works using this idea were those of Gotay and Bos [15], Arms, Cushman and Gotay [1], and Sjamaar and Lerman [28]. At this point it had become clear that the reduced space had dynamics, and that it could be described stratum by stratum using the Poisson bracket.[2]

Since then, it is now known that the reduced space is not only a topological space, but also has a differential structure, which is completely described by an algebra of smooth functions. These smooth functions are push-forwards of functions on the original space that are invariant under the group action. Such singular spaces are described naturally by the theory of subcartesian differential spaces, and in the case under consideration, *support dynamics as well* because the algebra of smooth functions has a Poisson structure. It is our view that satisfying the dual requirements of describing the analytic structure of the singularities of the reduced space *and* defining the reduced dynamics provides a powerful justification for our use of differential spaces.

A related development in the theory of constrained Hamiltonian systems with symmetry has been the reduction of nonholonomic constraints. The regular theory for transverse linear constraints was considered by Koiller [19], and extended to the nontransverse case by Bates and Śniatycki [4]. Regular reduction of nonlinear nonholonomic constraints was given by de Leon and de Diego in [13], and singular examples involving linear constraints were considered by Bates in [3]. The singular reduction of nonlinear nonholonomic constraints was given by Bates and Nester in [7]. What is notable here is that the formulation is once again in terms of invariant functions and the Poisson bracket, the wrinkle being that the Hamiltonian operator need no longer be an invariant function, and so the reduced dynamics is given by an *outer* Poisson morphism.

Of course, constraints in mechanics do not have to have anything to do with symmetry. There is a less mature, but somewhat parallel stream of development that tries to understand the nature of the constraints that show up in systems where the Lagrangian is degenerate in the sense that the Legendre transformation does not define a local diffeomorphism. This theory, inaugurated by Dirac in [14], (giving what is now called the Dirac constraint algorithm), describes a way to produce a Hamiltonian on a submanifold of the phase space. The constraint algorithm has been geometrized by Gotay, Nester and Hinds [16] and Lusanna [21]. However, the nature of such constraints in the Lagrangian is such that the initial data set, which is the subset of the original space on which the Lagrangian is defined actually

[2]It is our contention that this is about as far as the theory can be developed without the notion of differential spaces. The results of this stage of the development of singular reduction are completely described in the monograph of Ratiu and Ortega [26].