ON THE SPECTRA OF FERMIONIC SECOND QUANTIZATION OPERATORS

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We derive several formulae for the spectra of the second quantization operators in abstract fermionic Fock spaces.

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1. Introduction

Abstract theory of Fock spaces [1–4] provides powerful mathematical tools when one analyzes models of quantum field theory, the most promising physical theory which is expected to describe the fundamental interactions of elementary particles. This results from the fact that quantum filed theory deals with a quantum system with infinitely many degrees of freedom, including particles which may be created or annihilated, and that Fock spaces are furnished with suitable structure to describe particle creation or annihilation. In mathematical physics, two different types of Fock spaces, bosonic (or symmetric) Fock space and fermionic (or antisymmetric) Fock spaces, are considered, reflecting the fact that there are two different sorts of elementary particles in Nature—bosons and fermions.

In mathematical analyses of quantum theories, one of the most important problems includes to determine the spectra of various self-adjoint operators representing physical observables, especially, that of a Hamiltonian, which represents the total energy of the system under consideration. To each self-adjoint operator A acting in an underlying one-particle Hilbert space $\mathcal{H}$, bosonic or fermionic second quantization is defined as an operator which naturally "lifts" A up to the bosonic or fermionic Fock space over $\mathcal{H}$, respectively. In a bosonic Fock space, the spectra of second quantization operators were well investigated and useful formulae for them have been available. However, as far as we know, the corresponding useful formulae in a fermionic Fock space are still missing.

This situation is quite unsatisfactory to mathematical physicists since all matters in our world consist of fermions—quarks and leptons—and thus, in order to mathematically analyze a realistic model of quantum field theory, we have to treat not only bosonic but also fermonic second quantization operators. In such models, the fermionic second quantization operator gives a kinetic term of each fermion species, and total Hamiltonian is given by the sum of kinetic part (or free part) and interaction part. It does not seem to be possible to exactly "solve" the realistic models, and it is natural to treat the interaction part as a "small" perturbation of the free Hamiltonian. In these analyses, we have to know the detail structure of the unperturbed free Hamiltonian in advance, which is given by fermionic second quantization. The main motivation of the present work is to construct abstract and general formulae for spectra of fermonic second quantization operators, which are potentially applicable to all the quantum field models containing a fermion field operator.

Some problems in abstract fermionic Fock spaces can be analyzed in parallel with a bosonic case just by replacing symmetry with anti-symmetry, but other problems require completely new mathematical techniques which are essentially different from the ones used in the study of the bosonic case, and the identification of the spectrum of the fermionic second quantization operator is of the latter type. In a physical viewpoint, fermions are characterized by *Pauli's exclusion principle*, which asserts that one quantum state can be occupied by at most *one* fermion, whereas arbitrarily many bosons can occupy one quantum state at the same time. This exclusion principle prevents us from just mimicking the proof of bosonic case to obtain fermionic spectral formulae.

Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$ (we omit the subscript $\mathcal{H}$ if there will be no danger of confusion). For a linear operator T on $\mathcal{H}$, we denote its domain by $D(T)$. For a subspace $D \subset D(T)$, the symbol $T \upharpoonright D$ denotes the restriction of T to D. We denote by $\bar{T}$ the closure of T if T is closable. The spectrum (resp. the point spectrum) of T is denoted by $\sigma(T)$ (resp. $\sigma_{\mathrm{p}}(T)$). The symbol $\otimes^n \mathcal{H}$ (resp. $\wedge^n \mathcal{H}$) denotes the n-fold tensor product of $\mathcal{H}$ (resp. the n-fold antisymmetric tensor product). Let $\mathfrak{S}_n$ be the symmetric group of order n. The antisymmetrization operator $\mathcal{A}_n$ on $\otimes^n \mathcal{H}$ is defined to be

$$\mathcal{A}_n := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) U_\sigma,$$

where U_σ is a unitary operator on $\otimes^n \mathcal{H}$ such that

$$U_\sigma(\psi_1 \otimes \cdots \otimes \psi_n) = \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(n)}, \qquad \psi_j \in \mathcal{H}, \qquad j = 1, \ldots, n,$$

and $\mathrm{sgn}(\sigma)$ is the signature of the permutation $\sigma \in \mathfrak{S}_n$. Then, $\mathcal{A}_n$ is an orthogonal projection onto $\wedge^n \mathcal{H}$. The fermionic Fock space over $\mathcal{H}$ is defined by

$$\mathcal{F}_{\mathrm{f}}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{H} := \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \,\middle|\, \Psi^{(n)} \in \wedge^n \mathcal{H},\ \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2 < \infty \right\}.$$

For a densely defined closable operator A on $\mathcal{H}$ and $j = 1, \ldots, n$, we define a linear operator $\widetilde{A}_j$ on $\otimes^n \mathcal{H}$ by