ON THE ROTATION MATRIX IN MINKOWSKI SPACE-TIME

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In this paper, a Rodrigues-like formula is derived for 4×4 semi skew-symmetric real matrices in $\mathbb{E}_1^4$. For this purpose, we use the decomposition of a semi skew-symmetric matrix $A = \theta_1 A_1 + \theta_2 A_2$ by two unique semi skew-symmetric matrices A_1 and A_2 satisfying the properties $A_1 A_2 = 0$, $A_1^3 = A_1$ and $A_2^3 = -A_2$. Then, we find Lorentzian rotation matrices with semi skew-symmetric matrices by Rodrigues-like formula. Furthermore, we give a way to find the semi skew-symmetric matrix A for a given Lorentzian rotation matrix R such that $R = e^A$.

Keywords: Minkowski space-time, rotation matrix, Rodrigues formula.

MSC 2000: 15B10, 15A16, 53B30.

1. Introduction

Rotation matrices are very important matrices especially for computer sciences. For this reason, the generation of a rotation matrix becomes one of the most important problems for mathematicians. Generating a rotation matrix using a unit quaternion is given in the study [1] in a fairly elegant way. On the other hand, Rodrigues formula is a quite useful formula to generate the rotation matrix for a given rotation angle θ around a given rotation axis. This formula allows to compute e^A for a 3×3 skew-symmetric matrix A. If we take the skew-symmetric matrix as

$$A = \begin{bmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{bmatrix},$$

where $\mathbf{u} = (a, -b, c)$ is a unit vector, then we get the Rodrigues formula

$$R = e^{\theta A} = I + \sin\theta A + (1 - \cos\theta)\, A^2$$

by using the property $A^3 = -A$. The matrix R is the rotation matrix where $\mathbf{u}$ is the rotation axis and θ is the rotation angle. Unfortunately, there does not appear to be any simple way of obtaining a formula for e^A, where $A \in \mathrm{SO}(n, 1)$, [2]. A simple way of obtaining a formula for e^A can be given for $n = 3$, [3]. In the four-dimensional Euclidean space, a formula to find a rotation matrix using exponential map is derived in [4]. Also, a generalization of Rodrigues formula in the n-dimensional Euclidean space is given in the study [5]. Furthermore, the formulae for exponential of semi symmetric matrices of order 4 are deeply discussed in [6]. In [1], rotation matrices in Minkowski 3-space are generated with unit timelike split quaternions. Besides, the Rodrigues formula is used to obtain a rotation matrix in a Minkowski 3-space in [7]. But, the Rodrigues formula in the Minkowski 3-space changes whether the rotation axis is spacelike or timelike.

(i) If the rotation axis is timelike, $R = e^A = I + \sin\theta \ A + (1 - \cos\theta)\, A^2$,
(ii) If the rotation axis is spacelike, $R = e^A = I + \sinh\theta \ A + (\cosh\theta - 1)\, A^2$.

Also, a formula in semi-Euclidean space $\mathbb{E}_2^4$ is found in [6] by using the method given in the study [4]. But there is no such formula in Minkowski space-time. In this paper, a Rodrigues-like formula is derived for 4×4 semi skew-symmetric real matrices in $\mathbb{E}_1^4$. For this purpose, we use the decomposition of semi skew-symmetric matrix A as $A = \theta_1 A_1 + \theta_2 A_2$ by two semi skew-symmetric matrices A_1 and A_2 satisfying the properties $A_1 A_2 = 0$, $A_1^3 = A_1$ and $A_2^3 = -A_2$. First of all, we prove that A_1 and A_2 are uniquely obtained for a skew-symmetric matrix. Then, we find Lorentzian rotation matrices with skew-symmetric matrices by Rodrigues-like formula. Furthermore, we give a way to find the semi skew-symmetric matrix A for a given Lorentzian rotation matrix R such that $R = e^A$.

2. Preliminaries

Minkowski space-time is a four-dimensional Euclidean space which is provided with the Lorentzian inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{L}} = -u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

for vectors $\mathbf{u} = \left(u_1, u_1, u_3, u_4\right)$, $\mathbf{v} = (v_1, v_2, v_3, v_4)$ and is denoted by $\mathbb{E}_1^4$. We say that the vector $\mathbf{u}$ in $\mathbb{E}_1^4$ is called spacelike, lightlike (null) or timelike if $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}} > 0$, $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}} = 0$ or $\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}} < 0$, respectively. The norm of the vector $\mathbf{u} \in \mathbb{E}_1^4$ is defined by $\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}}|}$.

For any $R \in M_{4\times 4}(\mathbb{R})$, if $\langle R\mathbf{u}, R\mathbf{u} \rangle_{\mathbb{L}} = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{L}}$ for all vectors $\mathbf{u} \in \mathbb{E}_1^4$, then R is called a semi orthogonal matrix. That is, semi orthogonal matrices preserve the length of vectors in the Minkowski space-time and columns (or rows) of the semi orthogonal matrix form an orthonormal basis of $\mathbb{E}_1^4$. Moreover, R is a semi orthogonal matrix if and only if $I^* R^T I^* = R^{-1}$ where R^T is transpose of the