SCHMIDT-CORRELATED STATES, WEAK SCHMIDT DECOMPOSITION AND GENERALIZED BELL BASES RELATED TO HADAMARD MATRICES

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We study the mathematical structures and relations among some quantities in the theory of quantum entanglement, such as separability, weak Schmidt decompositions, Hadamard matrices etc. We provide an operational method to identify the Schmidt-correlated states by using weak Schmidt decomposition. We show that a mixed state is Schmidt-correlated if and only if its spectral decomposition consists of a set of pure eigenstates which can be simultaneously diagonalized in weak Schmidt decomposition, i.e. allowing for complex-valued diagonal entries. For such states, the separability is reduced to the orthogonality conditions of the vectors consisting of diagonal entries associated to the eigenstates; moreover, for a special subclass of these states this is surprisingly related to the so-called complex Hadamard matrices. Using the Hadamard matrices, we provide a variety of generalized maximal entangled Bell bases.

Keywords: quantum entanglement, Schmidt-correlated states, weak Schmidt decompositions, complex-valued simultaneous diagonalization, Hadamard matrices, generalized Bell bases.

1. Introduction

As one of the most striking features of quantum systems, quantum entanglement [1] plays crucial roles in quantum information processing [2] such as quantum computation, quantum teleportation, dense coding, quantum cryptographic schemes, quantum radar, entanglement swapping and remote states preparation. Nevertheless, many significant open problems in characterizing the entanglement of quantum systems still remain open.

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \simeq \mathbb{C}^n \otimes \mathbb{C}^n$ be a bipartite composite system. The mathematical problem consists in deriving separability criteria of mixed quantum states, and more generally, in quantifying their degree of entanglement. Such basic problems in the theory of quantum entanglement turn out to be surprisingly difficult, see e.g. [3] for a monographic treatment. Reasons for the difficulty are that the representations of a mixed state ρ as a (statistical) ensemble of pure states are not unique, and that the pure states in a representation in general cannot be simultaneously diagonalized in terms of suitable bases of $\mathcal{H}_A$ and $\mathcal{H}_B$ (Schmidt decomposition).

One of the main ideas of the paper is that one can weaken the requirement of simultaneous Schmidt diagonalization to a more general complex version, that is, the generalized Schmidt coefficients are allowed to be complex-valued, see Section 2. We call it the *weak Schmidt decomposition*. It was first introduced by [4] to study quantum states. In our applications, this concept will invoke the Hadamard matrices, a class of matrices that already have received considerable mathematical attention, though some basic problems still remain unresolved, see [5] for a survey.

We first recall some basic concepts in the theory of quantum entanglement. The entanglement of formation [6–9] and concurrence [10–13] are among the important measures to quantify the entanglement. However, due to the extremizations involved in the computation, only a few analytic formulae have been obtained for states like two-qubit ones [10, 14], isotropic states [15] and Werner states [16]. Instead of analytic formulae, some progress has been made toward the lower and upper bounds [17–26].

A mixed state ρ is called *Schmidt-correlated*, or *maximally correlated* [4, 27, 28], if there exists an orthonormal basis, $\{|e_j f_l\rangle\}_{j,l=1}^n$, of $\mathcal{H}$ such that

$$\rho = \sum_{j,l=1}^{n} C_{jl} |e_j f_j\rangle\langle e_l f_l|. \tag{1}$$

It is called maximally correlated since for any classical measurement on $\mathcal{H}_A$ or $\mathcal{H}_B$, Alice and Bob will always obtain the same results. One readily sees that Schmidt-correlated states are at most of rank n. It turns out that this class of states exhibits many excellent properties [4, 27, 29–33]. However, given a general state ρ written in the computational basis, any operational method to decide whether it is Schmidt-correlated is still missing in the literature. In this paper, we show that to find out whether a state is Schmidt-correlated it suffices to check whether its spectral decomposition consists of pure eigenstates which can be simultaneous diagonalized in weak Schmidt decomposition, see Theorem 3.1. Although the spectral