KLEIN–GORDON PARTICLE IN A ONE-DIMENSIONAL BOX WITH A MOVING WALL

O. HAMIDI and H. DEHGHAN

Department of Physics, Shahid Bahonar University of Kerman, Kerman, Iran
(e-mail: hamidi@mail.uk.ac.ir)

(*Received June 14, 2013 – Revised August 19, 2013*)

Relativistic treatment of a quantum mechanical massive particle in a one-dimensional box with one moving wall is presented using the Klein–Gordon equation. Despite the time dependence of the boundary, a separation of variables has been achieved for uniform wall motion by first transforming to a fixed boundary and subsequently introducing a new time coordinate. Exact solutions for both single variable differential equations have been obtained.

Keywords: moving boundary, separation of variables, Klein–Gordon equation, time-dependent normal modes.

PACS: 03.65.Pm

1. Introduction

The problem of a nonrelativistic massive particle in a one-dimensional box with fixed walls is discussed in every book on elementary quantum mechanics using the Schrödinger equation. The same problem is not so easy to solve if one of the walls is moving and is no longer separable. It is the quantum mechanical analog of the model proposed first by Fermi [1] in the context of cosmic ray acceleration and later quantum mechanically treated by Ulam [2]. This problem was originally solved for a uniformly moving wall in [3] and was later extended to wall motions satisfying the criterion $L^3\ddot{L} = \text{const}$ by a number of authors [4–9]. So far these are among the few exactly solvable quantum mechanical time dependent boundary problems.

In this work we have presented a relativistic treatment of a massive quantum mechanical particle in a one-dimensional box with one moving wall based on the Klein–Gordon equation which has also been recently addressed in [10]. Our solutions turn out to be the same as those in [10] but our approach is different and hinges on achieving a separation of variable, in the case of uniform wall motion, i.e. $L(t) = L_0 + vt$, by a suitable introduction of a time variable. As is explained in Section 2, the new time variable τ is constructed in such a way that it approaches the regular time t in the limit when the velocity of the moving wall v approaches zero and the ordinary Klein–Gordon equation results. It is shown that an infinite set of discreet orthogonal solutions exist.

2. Statement of the problem

Consider a Klein–Gordon particle inside a one-dimensional box with impenetrable boundaries at $x = 0$ and $x = L(t)$ described by the Klein–Gordon Equation (KGE)

$$\frac{1}{c^2}\frac{\partial^2\psi(x,t)}{dt^2} - \frac{\partial^2\psi(x,t)}{\partial x^2} + \mu^2\psi(x,t) = 0, \qquad \mu = \frac{m_0 c}{\hbar}. \tag{1}$$

In the nonrelativistic version of this problem, following [4], one turns the moving right wall into a fixed one by introducing a dimensionless spacial coordinate y defined by $y = x/L(t)$. Here in order to render the problem separable we further introduce a time coordinate by

$$t \to \tau = \tau(x,t) \tag{2}$$

whose explicit functional dependence will be given shortly in such a way that the resulting equation is separable for uniform wall motion. Note that the boundary at $x = L(t)$ is now fixed at the new coordinate $y = 1$. Expressing the space and time derivatives, $\partial/\partial x$ and $\partial/\partial t$ in terms of $\partial/\partial y$ and $\partial/\partial\tau$, and multiplying the resulting equation by $L^2(t)$, the KGE takes the form

$$\begin{aligned} L^2\left[\frac{1}{c^2}\left(\frac{\partial\tau}{\partial t}\right)^2 - \left(\frac{\partial\tau}{\partial x}\right)^2\right]\frac{\partial^2\Psi}{\partial\tau^2} - \left[1 - \frac{\dot{L}^2 y^2}{c^2}\right]\frac{\partial^2\Psi}{\partial y^2} \\ + L^2\left[\frac{1}{c^2}\frac{\partial^2\tau}{\partial t^2} - \frac{\partial^2\tau}{\partial x^2}\right]\frac{\partial\Psi}{\partial\tau} - \frac{1}{c^2}\left[L\ddot{L} - 2\dot{L}^2\right] y\frac{\partial\Psi}{\partial y} \\ - 2L\left[\frac{\dot{L}}{c^2} y\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial x}\right]\frac{\partial^2\Psi}{\partial\tau\partial y} + L^2\mu^2\Psi = 0, \end{aligned} \tag{3}$$

where $\Psi(y,\tau) = \psi(x,t)$ with (x,t) expressed in terms of (y,τ) using $y = x/L(t)$ and the inverse of relation in Eq. (2). The new time coordinates τ must be chosen such that in the (y,τ) coordinates the equation is separable. The necessary (but not sufficient) condition for this is that the coefficient of $\partial^2\Psi/\partial\tau\partial y$ should vanish. To this end one must have

$$\frac{1}{x}\frac{\partial\tau}{\partial x} = -\frac{\dot{L}}{c^2 L}\frac{\partial\tau}{\partial t}. \tag{4}$$

Inspection of Eq. (4) suggests a solution for $\tau(x,t)$ of the form

$$\tau(x,t) = G(x) + K(t). \tag{5}$$

Setting each side of Eq. (4) equal to $(-B)$ one then finds after simple integration of each side

$$\tau(x,t) = Bc^2\int_0^t dt'\frac{L(t')}{\dot{L}(t')} - \frac{Bx^2}{2}. \tag{6}$$

For a uniformly moving wall, $L(t) = L_0 + vt$, the integral in equation Eq. (6) can be computed and one obtains

$$\tau(x,t) = t + \frac{vt^2}{2L_0} - \frac{vx^2}{2L_0 c^2} = \frac{L^2 - L_0^2}{2vL_0} - \frac{vx^2}{2c^2 L_0}, \tag{7}$$