THE CORRESPONDENCE BETWEEN MUTUALLY UNBIASED BASES AND MUTUALLY ORTHOGONAL EXTRAORDINARY SUPERSQUARES

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We study the connection between mutually unbiased bases and mutually orthogonal extraordinary supersquares, a wider class of squares which does not contain only the Latin squares. We show that there are four types of complete sets of mutually orthogonal extraordinary supersquares for the dimension $d = 8$. We introduce the concept of physical striation and show that this is equivalent to the extraordinary supersquare. The general algorithm for obtaining the mutually unbiased bases and the physical striations is constructed and it is shown that the complete set of mutually unbiased physical striations is equivalent to the complete set of mutually orthogonal extraordinary supersquares. We apply the algorithm to two examples: one for two-qubit systems ($d = 4$) and one for three-qubit systems ($d = 8$), by using the Type II complete sets of mutually orthogonal extraordinary supersquares of order 8.

Keywords: Latin squares, mutually unbiased bases.

1. Introduction

The mutually unbiased bases are widely used in many protocols of quantum information processing: quantum tomography [1–3], quantum cryptography [4], discrete Wigner function [5, 6], understanding the complementarity [7–10], quantum teleportation [11], quantum error correction codes [12], or the mean king's problem [13], just to enumerate a few of them. Two bases $\{|\,\psi_j\,\rangle\}$ and $\{|\,\phi_k\,\rangle\}$ are called mutually unbiased if for any two vectors one has $|\langle\,\psi_j\,|\,\phi_k\,\rangle|^2 = 1/d$, where d is the dimension of the Hilbert space [1]. The maximal number of mutually unbiased bases (MUBs) can be at most $d+1$ [14] and this value can be reached in the case when the dimension of the space is a prime or a power of prime [15].

MUBs can be constructed using different methods. One method consists in obtaining $d+1$ classes of $d-1$ commuting operators, whose eigenvectors represent the MUBs. These special operators are called mutually unbiased operators [16].

Wootters [17] and Gibbons et al. [6] proposed a geometrical approach that is based on the correspondence of the MUBs to the so-called discrete phase space. The phase space of a d-level system (qubit) is a $d \times d$ lattice, whose coordinates are elements of the finite Galois field $\mathbb{F}_d$. A state is associated to a line in the discrete phase space and the set of parallel lines is called a striation [17]. The line passing through the origin is called ray. It turns out that the MUBs are determined by the bases associated with each striation. A detailed review by Durt et al. presents different constructions of MUBs as well as their applications [18].

Recently, the connection between magic and Latin squares with quantum information theory has been investigated. The quantum game based on a magic square allows two observers Alice and Bob to share an entangled quantum state [19, 20]. Also an interesting application of Latin squares is quantum teleportation, where generalized Bell states are given in terms of Latin squares [21].

Similarities between MUBs and mutually orthogonal Latin squares have been analyzed during the last years [17–30]. In [17, 23], one considers the case when the two striations with vertical and horizontal lines are present among the set of mutually orthogonal striations. This fact leads only to a special class of MUBs, the one which contains the eigenvectors of tensor products of the Pauli operators X and the identity, Z and the identity, or their combinations. The associated striations of MUBs are Latin squares. In [24], a set of Latin operators is constructed, whose eigenvectors form a complete set of MUBs, therefore the associated striations are Latin squares. We should mention that the connection is done only with the Latin squares (also the squares with vertical and horizontal lines are considered) and not with a wider class.

There are at least two different ways of defining the problem of construction of MUBs:

A) Generation of the MU operators. This ends the problem and no connection to Latin squares is done.

B) Generation of both the MU operators and the striations associated to each basis. There are discussions on the link between striations and Latin squares. The striations are relevant if we want to compute the discrete Wigner function, since we need to know the expression of each line of all the striations.

In this paper we analyze the problem (B), by trying to answer to three questions. Suppose that one starts with the ray denoted by the number 1 as below in the square on Fig. 1:

?	?	?	**1**
?	?	**1**	?
?	**1**	?	?
1	?	?	?

Fig. 1. What properties should fulfill a striation associated to MUB?