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Reconstruction of coefficients of higher order nonlinear wave equations by measuring solitary waves

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HIGHLIGHTS

- Inverse problems for 5th and 6th order nonlinear wave equations are studied.
- The problems use measurements of characteristics of solitary waves.
- The uniqueness of solutions is shown and direct algorithms are deduced.

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ABSTRACT

Inverse problems to recover coefficients of the fifth order KdV equation, sixth order generalized Boussinesq equation (SGBE) and two sixth order equations occurring in the dynamics of multiscale microstructure are considered. It is proposed to use characteristics of solitary waves for solving the inverse problems. The uniqueness of the solutions is shown and solution algorithms are provided. Several numerical examples are presented.

1. Introduction

A general method to reconstruct coefficients of a partial differential equation (PDE) from measured states consists in the minimization of a proper cost functional. This procedure is often very time-consuming due to the necessity to incorporate global search techniques and is even more involved in the case of a nonlinear PDE of higher order, because the latter one has to be solved with high accuracy in each step of the minimization. Therefore, search for more direct and effective solution algorithms is motivated. To this end, simple waveforms may be useful.

In nonlinear dispersive media (e.g. shallow water, plasma, microstructured materials) solitary waves occur under proper conditions. This has been proved mathematically and observed experimentally [1–5]. Usage of solitary waves simplifies the inverse problem, because a PDE is replaced by an ordinary differential equation (ODE). Moreover, in particular cases even more effective direct algorithms may be found instead of the cost functional minimization.

A novel method for solving the inverse problems was proposed in [6,7] in order to recover coefficients of a 4th order equation and a coupled system that govern the motion of the microstructured material of Mindlin type. This method is based on measuring the characteristics of solitary waves and in such a way data related to nonlinearities at both macro- and

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microlevel as well as to dispersion are obtained. The method was generalized to periodic waves in [8]. In this paper also the cost functional minimization technique was compared with a direct usage of characteristics of solitary and periodic waves for the solution of inverse problems. The results convince in the efficiency of the latter method.

The class of mathematical models for describing the microstructured solids is certainly wider than analyzed in [6–8].

In the present paper we generalize this method to higher, namely 5th and 6th order nonlinear dispersive equations. More precisely, we consider the fifth order KdV equation, the sixth order generalized Boussinesq equation and two sixth order equations that occur in the theory of multiscale microstructure. All these equations can be reduced to a common canonical equation for the solitary wave, hence can be analyzed in the same framework.

We pose inverse problems to recover coefficients of these equations using amplitudes and lengths of solitary waves measured at given levels. The main tasks are to show the uniqueness of the solutions of these problems and find effective solution algorithms. A novel approach is proposed that combines analytical arguments and a numerical verification of the hypotheses that lie under those arguments to achieve these aims. Finally, the solution procedure is illustrated by numerical examples and the sensitivity with respect to errors of the data is discussed.

2. The 5th order KdV equation. Canonical equation

Let us consider the 5th order KdV equation

$$v_t + \alpha_1 (v^2)_x + \alpha_2 v_{xxx} - \alpha_3 v_{xxxxx} = 0. \tag{1}$$

This model occurs in many applications, e.g. capillary–gravity water waves, chains of coupled oscillators, magneto-acoustic waves in plasma [9–11]. The coefficients α_i are related to properties of the physical medium under consideration. Supplement with the fifth order term enables to overcome the short-wave instability that occurs in the usual KdV model. Inverse problems may be used to verify the relevance of this model for particular media. Namely, in case the solutions of the inverse problem corresponding to different data approximately coincide, the model is relevant, otherwise not.

The solitary wave is a solution of (1) satisfying the conditions

$$\begin{aligned} &v(x,t) = w(x-ct), \quad c - \text{constant}, \\ &w^{(j)}(\xi) \to 0 \quad \text{as } |\xi| \to \infty,\ j = 1,\dots,4. \end{aligned} \tag{2}$$

Parameter c is the velocity. Inserting $v(x,t) = w(x-ct)$ into (1) we get the following equation for the solitary wave:

$$-cw' + \alpha_1 (w^2)' + \alpha_2 w''' - \alpha_3 w^{V} = 0.$$

Integration yields

$$-cw + \alpha_1 (w^2) + \alpha_2 w'' - \alpha_3 w^{IV} = 0.$$

Let us go over to a canonical form introducing new variables

$$w(\xi) = s\Phi(\eta) \quad \text{with } \eta = \sigma\xi, \tag{3}$$

where s and σ are constants defined by the following relations via α_i:

$$\sigma^2 = \frac{\alpha_2}{\alpha_3}, \qquad s = \frac{\alpha_3}{\alpha_1}\sigma^4. \tag{4}$$

In this connection we assume that $\alpha_2\alpha_3 > 0$. Introducing the additional parameter

$$\vartheta = \frac{c}{\alpha_3\sigma^4}, \tag{5}$$

the equation for Φ takes the form of the following one-parametric 4th order ODE:

$$\Phi^{IV} - \Phi'' - \Phi^2 + \vartheta\Phi = 0. \tag{6}$$

The solitary waves of (1) are obtained by rescaling of localized solutions (homoclinic orbits) of the canonical equation (6). It is known that (6) may have one- and multi-pulse localized solutions [12]. For the parameter ϑ in the interval $0 < \vartheta \le \frac{1}{4}$ unique one-pulse solution exists [13]. (The uniqueness is understood to the accuracy of the constant shift of the argument.) The solution is monotonic, i.e. the inequalities $\Phi' > 0$ if $\eta < 0$ and $\Phi' < 0$ if $\eta > 0$ hold for the centered at $\eta = 0$ solution. In case $\vartheta > \frac{1}{4}$ also unique one-pulse solution exists [1,13], but it is non-monotonic (has slightly oscillating tails). An analytical formula for the one-pulse solution is known only in the case $\vartheta = \frac{36}{169} \in (0, \frac{1}{4})$. Then

$$\Phi(\eta) = \frac{105}{338}\,\mathrm{sech}^4\left(\frac{\eta}{2\sqrt{13}}\right).$$

In case $\vartheta > \frac{1}{4}$ also infinitely many multi-pulse localized solutions occur and in case $\vartheta \le 0$ no one-pulse solutions exist [1,13,12]. The condition $\vartheta > 0$ for the existence of one-pulse solitary wave in terms of the original parameters and velocity takes the form

$$\frac{c\alpha_3}{\alpha_2^2} > 0. \tag{7}$$