Wave Motion 52 (2015) 103–119

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Doppler effects of an oscillating line source in shear flow with a free surface

Peder A. Tyvand, Mikkel Elle Lepperød *

Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, P.O. Box 5003, 1432 Ås, Norway

HIGHLIGHTS

- Water-wave radiation with uniform vorticity is represented by a Green function: the 2D oscillating source.
- Nonzero surface velocity gives Doppler effects and at most four radiated waves.
- Wave motion co-rotating with the vorticity is easier to generate than counter-rotating wave motion.
- Wave cut-offs occur, and there is resonance with two waves merging at zero group velocity.

ARTICLE INFO

Article history:
Received 26 May 2014
Received in revised form 13 September 2014
Accepted 17 September 2014
Available online 28 September 2014

Keywords:
Doppler effects
Green function
Oscillating source
Shear flow
Water waves

ABSTRACT

The linearised water-wave radiation problem for the oscillating 2D submerged source in an inviscid shear flow with a free surface is investigated analytically. There is a nonzero surface velocity. The depth is infinite and the vorticity is uniform. The amplitudes radiated from the source are calculated analytically. Due to Doppler effects, there may be up to four different emitted waves, and there is resonance with zero group velocity and infinite amplitude.

1. Introduction

The submerged oscillatory source is an elementary solution for linearised water waves governed by Laplace's equation. Oscillatory sources are Green functions that satisfy the linearised free-surface condition and the radiation conditions at infinity. The first mathematical solutions were given by Kochin [1]; see the review article by Wehausen and Laitone [2].

When the flow is uniform, there is resonance with zero group velocity at the critical value 1/4 for the dimensionless frequency of the source. The wave amplitude at resonance is infinite according to linear theory. A submerged oscillating 2D body corresponds to an integrated source distribution, and one would expect infinite wave amplitude at resonance. However, [3] showed that the wave amplitude radiated from an oscillating circular cylinder has a finite peak at resonance.

Resonance for water wave radiation in the presence of a basic flow is linked to the existence of Doppler effects. Tyvand and Lepperød [4] calculated the radiated waves from a submerged 2D source in a shear flow where the velocity is zero at the undisturbed free surface. With zero surface velocity, there is no Doppler effect and no resonance. There are at most two radiated waves, one upstream wave and one downstream wave. When the angular frequency of the source is reduced below the vorticity, there is a cut-off where the downstream wave disappears abruptly.

* Corresponding author. Tel.: +47 922 16 758.
E-mail address: lep.mik@gmail.com (M.E. Lepperød).

http://dx.doi.org/10.1016/j.wavemoti.2014.09.005

In the present paper we take our previous analysis further by including a nonzero surface velocity. Thus the shear flow distribution is general, within the constraints of unidirectional flow with uniform vorticity. The problem becomes more complicated, as there are Doppler effects that make resonance possible. There will be up to four different emitted waves, and each of these will have different frequencies seen from an observer that is at rest with the free surface.

Vorticity is not included in the classical theory of submerged sources [2]. All 3D waves with vorticity will involve vortex stretching. For 2D flows with uniform vorticity, the methods of potential theory apply. We will solve the radiation problem for an oscillating line source in shear flow with uniform vorticity. We expect that it will be easier to generate waves where particles rotate with the vorticity, compared to waves with particles counter-rotating. Tyvand and Lepperød [4] found that a criterion for wave generation is that a particle rotates sufficiently quick to overcome an opposite rotation of the shear flow.

Miles [5] took vorticity into account to explain the generation of ocean waves by wind stress. The study of steady nonlinear water waves with vorticity was started already by Gerstner [6]. However, this early solution is valid only for a particular exponential distribution of the vorticity. More recent papers on steady rotational waves with finite amplitude have concentrated on the case of uniform vorticity. Simmen and Saffman [7] studied nonlinear waves at infinite depth. Teles da Silva and Peregrine [8] studied nonlinear waves at constant depth. A number of other papers have been written on steady nonlinear water waves with uniform vorticity. Baddour and Song [9] developed a second-order theory of steady waves that is valid for more general vorticity distributions.

The linear theory of time-dependent water waves with vorticity is still under development. Brevik [10] explored the possibility of stopping linearised gravity waves by means of shear flows. Jonsson et al. [11] and White [12] developed the concept of wave action on shear flows. Brevik and Sollie [13] discussed the energy flux. Two pioneering studies for 3D linearised water waves on shear flows have recently been presented by Ellingsen [14,15]: first the steady ship wave problem [14], generalising Lord Kelvin's classical work. Secondly the linearised Cauchy–Poisson problem for the time evolution of initial disturbances [15].

The literature on unsteady nonlinear water waves with vorticity is limited. A significant contribution is the work by Teles da Silva and Peregrine [16] on undular bores with uniform vorticity. Vorticity is important in the interactions between waves and currents in the ocean [17], so this is a challenging field for further investigations.

2. Mathematical model

We consider an inviscid and incompressible fluid in a steady shear flow along a horizontal x axis. The flow is two-dimensional in the x, y plane, and it is driven by a fixed oscillating line source located at a depth D. The semi-infinite fluid has a free surface subject to constant atmospheric pressure. Cartesian coordinates x, y are introduced, where the y axis is directed upwards in the gravity field and $y = 0$ represents the undisturbed free surface. The gravitational acceleration is g, and ρ denotes the constant fluid density. The surface elevation is denoted by $\eta(x, t)$, and the overall problem is sketched in Fig. 1.

We assume infinite fluid depth. There is a basic horizontal shear flow $U(y)$ in the x direction

$$U(y) = U_0 + \Omega y, \quad y < 0, \tag{1}$$

where the surface velocity is defined nonnegative ($U_0 \geq 0$). There is a uniform vorticity Ω, which is the velocity gradient of the steady shear flow. The vorticity is defined positive in the clockwise direction and negative in the counter-clockwise direction. Here we make an opposite sign convention for Ω compared with our previous paper [4]: we will now let a positive Ω represent a vorticity that is co-rotating with the particle motion of downstream waves.

The total velocity field $\mathbf{v}$ is the sum of an unsteady potential flow plus the steady shear flow

$$\mathbf{v} = \nabla\Phi + (U_0 + \Omega y)\mathbf{i}, \tag{2}$$

where $\Phi(x, y, t)$ is the velocity potential, and $\mathbf{i}$ is the horizontal unit vector.

Conservation of mass gives Laplace's equation

$$\nabla^2\Phi = 0, \tag{3}$$

valid in the fluid outside the source point $(x, y) = (0, -D)$.

Euler's equation of motion can be written

$$\nabla\left(\frac{p}{\rho} + \frac{\partial\Phi}{\partial t} + U_0\frac{\partial\Phi}{\partial x} + \frac{1}{2}|\nabla\Phi|^2 + gy\right) = -\Omega y\nabla\frac{\partial\Phi}{\partial x} - \Omega\frac{\partial\Phi}{\partial y}\mathbf{i}. \tag{4}$$

We recognise the terms from the Bernoulli equation for irrotational flow. There are two extra terms due to vorticity, and one due to the basic flow.

The linearised kinematic free-surface condition is

$$\frac{\partial\eta}{\partial t} + U_0\frac{\partial\eta}{\partial x} = \frac{\partial\Phi}{\partial y}, \quad y = 0. \tag{5}$$

The dynamic condition is that the pressure is constant at the free surface, since we neglect surface tension. This implies zero tangential derivative of the pressure. From the tangential component of Eq. (4) we find the linearised dynamic free-surface condition

$$\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial t} + U_0\frac{\partial\Phi}{\partial x} + g\eta\right) = -\Omega\frac{\partial\Phi}{\partial y}, \quad y = 0. \tag{6}$$