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# Numerical solution of the Benjamin equation

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## HIGHLIGHTS

- A novel hybrid scheme for the Benjamin equation is constructed.
- Accuracy and stability properties are shown.
- Evolution properties are validated with solitary wave simulations.
- Collisions and stability properties of solitary waves are studied.

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## ABSTRACT

In this paper we consider the Benjamin equation, a partial differential equation that models one-way propagation of long internal waves of small amplitude along the interface of two fluid layers under the effects of gravity and surface tension. We solve the periodic initial-value problem for the Benjamin equation numerically by a new fully discrete hybrid finite-element/spectral scheme, which we first validate by pinning down its accuracy and stability properties. After testing the evolution properties of the scheme in a study of propagation of single- and multi-pulse solitary waves of the Benjamin equation, we use it in an exploratory mode to illuminate phenomena such as overtaking collisions of solitary waves, and the stability of single-pulse, multi-pulse and 'depression' solitary waves.



## 1. Introduction

In this paper we will consider the *Benjamin equation*

$$u_t + \alpha u_x + \beta u u_x - \gamma \mathcal{H} u_{xx} - \delta u_{xxx} = 0, \tag{1.1}$$

where $u = u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, $\alpha, \beta, \gamma, \delta$ are positive constants, and $\mathcal{H}$ denotes the Hilbert transform defined on the real line as

$$\mathcal{H} f(x) := \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\, dy$$

or through its Fourier transform as

$$\widehat{\mathcal{H} f}(k) = -\mathrm{i}\, \mathrm{sign}(k) \widehat{f}(k), \quad k \in \mathbb{R}.$$

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The Benjamin equation, cf. [1–3], is a model for *internal* waves propagating under the effect of gravity and surface tension in the positive $x$-direction along the interface of a two-dimensional system of two homogeneous layers of incompressible, inviscid fluids consisting at rest of a thin layer of fluid 1 of depth $d_1$ and density $\rho_1$ lying above a layer of fluid 2 of very large depth $d_2 \gg d_1$ and density $\rho_2 > \rho_1$. The upper layer is bounded above by a horizontal 'rigid lid' and the lower layer is bounded below by an impermeable horizontal bottom.

It is further assumed that the following physical regime of interest is to be modelled: Let $a$ be a typical amplitude and $\lambda$ a typical wavelength of the interfacial wave. The parameters $\epsilon = a/d_1$ and $\mu = d_1^2/\lambda^2$ are assumed to be small and satisfy $\mu \sim \epsilon^2 \ll 1$; it is also assumed that capillarity effects along the interface are not negligible. Under these assumption (1.1) was derived in [1] from the two-dimensional, two-layer Euler equations in the presence of interface surface tension by dispersion relation arguments. The variables in (1.1) are nondimensional and scaled, and the coefficients are given by

$$\alpha = \sqrt{\frac{\rho_2 - \rho_1}{\rho_1}}, \qquad \beta = \frac{3}{2}\alpha\epsilon, \qquad \gamma = \frac{1}{2}\alpha\sqrt{\mu}\frac{\rho_2}{\rho_1}, \qquad \delta = \frac{\alpha T}{2g\lambda^2(\rho_2 - \rho_1)},$$

where $T$ is the interfacial surface tension and $g$ the acceleration of gravity. The variables $x$ and $t$ are proportional to distance along the channel and time, respectively, and $u(x, t)$ denotes the downward vertical displacement of the interface from its level of rest at $(x, t)$. The interfacial surface tension $T$ is assumed to be much larger than $g(\rho_2-\rho_1)d_1^2$. (For a further discussion of the physical regime of validity of (1.1) cf. [3].) Note that if the parameter $\delta$ is taken equal to zero, (1.1) reduces to the Benjamin–Ono (BO) equation, [4,5], while, if we put $\gamma = 0$ we obtain the KdV equation with negative dispersion coefficient.

It is well known, cf. [1], that sufficiently smooth solutions of (1.1) that vanish suitably at infinity preserve the functionals

$$m(u) = \int_{-\infty}^{\infty} u dx, \tag{1.2}$$

$$I(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx, \tag{1.3}$$

$$E(u) = \int_{-\infty}^{\infty} \left(\frac{\beta}{6}u^3 - \frac{1}{2}\gamma u \mathcal{H} u_x + \frac{1}{2}\delta u_x^2\right) dx. \tag{1.4}$$

Global well-posedness in $L^2$ for the Cauchy problem and also for the periodic initial-value problem for (1.1) was established in [6].

In this paper we will study (1.1) numerically, paying particular attention to properties of its *solitary-wave* solutions. These are travelling-wave solutions of the form $u(x, t) = \varphi(x - c_s t)$, $c_s > 0$, such that $\varphi$ and its derivatives tend to zero as $\xi = x - c_s t$ approaches $\pm\infty$. Substituting this expression in (1.1) and integrating once we obtain

$$(\alpha - c_s)\varphi + \frac{\beta}{2}\varphi^2 - \gamma \mathrm{H}\varphi - \delta\varphi'' = 0, \tag{1.5}$$

where $' = d/d\xi$, and the operator H is defined by $\mathrm{H} := \mathcal{H}\partial_x$, i. e. by $\widehat{\mathrm{H}f}(k) = |k|\widehat{f}(k)$, $k \in \mathbb{R}$. We will assume that $\alpha - c_s > 0$.

If we perform the change of variables

$$\varphi(\xi) = -\frac{2(\alpha - c_s)}{\beta}\psi(z), \qquad z = \sqrt{\frac{\alpha - c_s}{\delta}}\xi,$$

in (1.5), we see that the solitary-wave profile $\psi(z)$ satisfies the ordinary differential equation (o.d.e.)

$$\psi - 2\tilde{\gamma}\mathrm{H}\psi - \psi_{zz} - \psi^2 = 0, \quad z \in \mathbb{R}, \tag{1.6}$$

where

$$\tilde{\gamma} = \frac{\gamma}{2\sqrt{\delta(\alpha - c_s)}}. \tag{1.7}$$

This change of variables and the resulting Eq. (1.6) was used in [1,2], and [3]. (In these references $\tilde{\gamma}$ is denoted by $\gamma$.) In his papers Benjamin showed that for each $\tilde{\gamma} \in [0, 1)$, there exists a solution $\psi$ of (1.6) which is an even function of $z$ with $\psi(0) = \max_{z\in\mathbb{R}} \psi(z) > 0$. He also argued by formal asymptotics that for each $\tilde{\gamma} \in [0, 1)$ there is a bounded interval centred at $z = 0$, in which $\psi$ oscillates (with the number of oscillations increasing as $\tilde{\gamma}$ approaches 1), while outside this interval he concluded in [2] that $|\psi|$ decays like $1/z^2$. In addition, in the same paper he outlined an orbital stability theory for these solitary waves for small $\tilde{\gamma}$. In [3] a complete theory of existence and orbital stability of the solitary waves for small $\tilde{\gamma}$ was presented. Further issues of existence and rigorous asymptotics of the solitary waves of (1.1) and related equations were explored in [7]. In [8] concentration compactness arguments were used to establish existence and a weaker version of stability of the solitary waves of (1.1) for $0 < \tilde{\gamma} < 1$.

In this paper we will employ the solitary-wave equation in the form (1.5). As a result, normally the solitary waves will have negative maximum excursions from their level of rest.

Since explicit formulas for the solitary waves of the Benjamin equation are not known (except when one of $\gamma$ or $\delta$ is set equal to zero), one must resort to approximate techniques for their construction. The presence of the nonlocal terms in (1.1) and (1.5), which have a handy Fourier representation in the periodic case as well, naturally suggests using spectral-type methods for approximating their solutions. The preceding discussion of the Benjamin equation applies to its associated