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Weakly nonlinear wave interactions in multi-degree of freedom periodic structures

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HIGHLIGHTS

- We analyze elastic wave interactions in nonlinear, periodic materials.
- We consider multi-degree of freedom and multidimensional systems.
- Wave interactions allow for novel control of wave direction and group velocity.
- Wave interactions suggest superprism, focusing, and multiplexing applications.

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ABSTRACT

This work presents a multiple time scales perturbation analysis for analyzing weakly nonlinear wave interactions in multi-degree of freedom periodic structures. The perturbation analysis is broadly applicable to (discretized) periodic systems in any dimensional space and with a wide range of constitutive nonlinearities. Specific emphasis is placed on cubic nonlinearity, as dispersion shifts typically arise from the cubic components in nonlinear restoring forces. The procedure is first presented in general. Then, application to the diatomic chain and monoatomic two-dimensional lattice demonstrates, individually, the treatment of multiple degree of freedom systems and higher dimensional spaces. The dispersion relations are modified by weakly nonlinear wave interactions and lead to additional opportunities to control wave propagation direction, band gap size, and group velocity. Numerical simulations validate the expected dispersion shifts. An amplitude-tunable focus device demonstrates the viability of utilizing dynamically-introduced dispersion to produce beam steering that may, ultimately, lead to a phononic superprism effect as well as multiplexing/demultiplexing behavior.

1. Introduction

Periodic structures exhibit dispersive wave propagation whereby the phase and group velocities are frequency-dependent. In addition to wave dispersion, periodic structures often exhibit frequency bands where elastic wave propagation is prohibited in one or more directions. Such frequency bands are termed bandgaps. Mechanical systems designed to explicitly utilize these unique filtering properties to control the propagation of elastic energy are usually termed phononic crystals [1]. Phononic crystal devices such as wave guides and resonators are generally formed from the strategic periodic arrangement of unit cells [2,3]. Crystal lattice planes and graphene sheets also exhibit lattice periodicity where nonlinear

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restoring forces may arise from inter-atomic attraction and repulsion. Waves in *linear* elastic periodic structures are generally termed Bloch waves (or Floquet–Bloch waves). Frequency-domain analysis for such structures results in a Bloch wave dispersion relation which can be used to locate bandgaps and energy propagation directions (group velocity). Thus, the dispersion relation is a critical tool in designing and analyzing phononic crystals and other periodic structures.

A complication arises when the governing wave equation contains nonlinear terms that may result from constitutive laws or finite displacements (i.e., material or geometric nonlinearity). In these situations, traditional Bloch wave analyses are not strictly applicable. Despite this complication, numerical and experimental evidence suggests that Bloch-like waves propagate in periodic structures with weak nonlinearities [4,5]. Unlike Bloch wave propagation in a linear system, the dispersive properties vary according to the local amplitude [6]. Thus, the location of band gaps, group velocity magnitude, and even the direction of energy propagation are all amplitude-dependent [7,8]. The same dispersion amplitude-dependence appears in crystal lattices where semi-empirical interatomic potentials (e.g., Born–Meyer, Lennard-Jones) contribute nonlinear stiffness terms [9].

The concept of a wave's amplitude influencing its own propagation characteristics is often referred to as *self-action* [10]. The effects of self-action have been predominantly investigated in nonlinear optics and electromagnetics literature where they have been linked to soliton solutions, harmonic generation, self-focusing and defocusing, phase modulation, and pulse compression [11–13]. Most efforts consider the effects of wave propagation in homogeneous material, although some have investigated self-action in periodic optical and elastic structures [13–16]. Self-action in nonlinear elastic structures, such as phononic crystals and atomic lattices, has recently been investigated using a perturbation approach designed specifically for weakly nonlinear Bloch wave propagation [17,18]. However, because solutions to nonlinear wave equations are amplitude-dependent, these self-action analysis methods often break down when the frequency content of the solution contains more than one dominant component.

Multi-harmonic excitations result in nonlinear wave interactions which give rise to sum-and-difference frequencies, cross-phase modulation, and frequency conversion among other effects [11,12]. Thus, the introduction of additional waves into a phononic device can control or alter the behavior of the system beyond the effects of self-action. Nonlinear wave–wave interactions and their effect on dispersion in the monoatomic chain were considered in [19]. It was shown that the each primary harmonic in the wave solution obeys a fundamentally different dispersion curve. The Bloch wave-based multiple scales perturbation analysis method developed in [19] was validated through extensive numerical simulation. In the monoatomic chain, the propagation velocity depended on the wavenumber and amplitude (in addition to physical system parameters).

Nonlinear wave interactions in two-dimensions and multi-degree of freedom (DOF) unit cells offer a fundamentally different perspective for viewing tunability. Wave propagation in two dimensional systems exhibit directionality in addition to other dispersion behaviors exhibited by one dimensional systems (1D). Directionality in 2D structures introduces conceptually new opportunities such as wave beaming, spatial filtering, and imaging. Wave beaming in the two-dimensional beam grillage was considered by Langley et al. in [20], while others have considered wave beaming in two-dimensional cellular structures [21–24]. These more complicated spatial systems provide new tunable parameters: additional wave vectors, amplitudes, and non-trivial wave modes. The idea that nonlinear wave interactions can enhance traits in two-dimensional (2D) periodic systems has been explored more recently in the photonic crystal community. Panoiu et al. utilized the Kerr nonlinearity with a pump/control wave to enhance the "superprism" effect, whereby the direction of propagation in the photonic crystal is extremely sensitive to the wavelength and angle of incidence [25].

Robust tools and analysis methods for nonlinear wave–wave interactions in more general elastic periodic structures are needed. Thus, the aim of the present research is to continue the development of the multiple scales perturbation presented in [19] with the intent of realizing wave-tunable dispersion in multi-DOF unit cells, to include two-dimensional structures. Previous multiple scales analysis methods have been restricted to single DOF systems with only a primary harmonic. We circumvent these restrictions by providing a more general multiple scales analysis framework. After developing the analysis method, it is applied to the diatomic chain with two DOF per unit cell to illustrate wave mode dependencies. Then, the 2D anharmonic lattice is considered in order to explore wave directionality and tunability that results from nonlinear wave–wave interactions for co-propagating, orthogonal, and oblique wave–wave interactions. Finally, a tunable-focus concept device is presented as a potential application of nonlinear wave–wave interactions in phononic crystals.

2. Multiple scales analysis for wave interactions

We first review a Bloch-informed method of multiple scales [26–28] for analyzing wave propagation in periodic structures, and then specialize the presented approach to analyze wave interactions in materials whose unit cells contain multiple degrees of freedom and/or dimensions. Consider a general three-dimensional unit cell located at a lattice point defined by the integers (p, q, r) which are associated with lattice vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. These indices (p, q, r) take the value 0 when referring to the central unit cell under consideration, and ± 1 when referring to adjacent unit cells. The governing equations for a single unit cell are first discretized (e.g., via finite difference approaches, finite element approaches, etc.) into the form

$$\mathbf{M}\ddot{\mathbf{u}} + \sum_{(p,q,r)} \mathbf{K}_{(p,q,r)}\mathbf{u}_{(p,q,r)} + \varepsilon \mathbf{f}^{NL}\left(\mathbf{u}_{(p,q,r)}\right) = \mathbf{0}, \tag{2.1}$$