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Numerical and asymptotic solutions of generalised Burgers' equation

J.M. Schofield *, P.W. Hammerton

School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK

HIGHLIGHTS

- We generate a numerical scheme to solve Burgers' equation using a variable mesh scheme.
- We validate asymptotic prediction for shock location and width for small times.
- Show partial breakdown in spherical case from a tanh shock to an erf shock.
- Offers validation of Enflo's old age form for cylindrical case.

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ABSTRACT

The generalised Burgers' equation models the nonlinear evolution of acoustic disturbances subject to thermoviscous dissipation. When thermoviscous effects are small, asymptotic analysis predicts the development of a narrow shock region, which widens, leading eventually to a shock-free linear decay regime. The exact nature of the evolution differs subtly depending upon whether plane waves are considered, or cylindrical or spherical spreading waves. This paper focuses on the differences in asymptotic shock structure and validates the asymptotic predictions by comparison with numerical solutions. Precise expressions for the shock width and shock location are also obtained.

1. Introduction

When studying the propagation of sonic booms through the atmosphere, the shock amplitude and shock width (or equivalently shock rise-time) are the quantities of most interest, especially when considering the annoyance and potential damage associated with such shocks. Initially, model equations with one spatial dimension were studied taking account of the combined effect of nonlinear steepening and thermoviscous diffusion [1]. This simple model was then extended to take account of geometric spreading [2] with justification that these model governing equations form a rational approximation to the full equations of fluid motion. Further studies include the effect of atmospheric stratification [3], other dissipation mechanisms such as molecular relaxation [4,5] or combinations of all these effects [6,7]. Alongside these investigations, numerical studies typically involved either direct solution of model equations [8,9] or separation of nonlinear terms from linear frequency-dependent terms [10–13].

For the physical parameter ranges of most interest, the shock width is small compared to the overall wavelength and the need for a good estimate of the shock width leads to the need for fine spatial resolution. Moreover, it is found that in some regimes the shock width is sensitive to small changes in material parameters. A numerical solution, while identifying such changes, does little to highlight which physical processes are of key importance. For these reasons, asymptotic analysis

* Corresponding author. Tel.: +44 7969775683.
E-mail address: j.schofield@uea.ac.uk (J.M. Schofield).

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making use of the disparity in spatial scales is a useful approach. In the case of wave disturbances spreading with cylindrical or spherical symmetry, Crighton and Scott [14] chart the variation of the shock width, along with the internal shock structure, up to a stage at which the evolution is linear and shock-free (the so-called old-age solution). Such an approach, while very intricate, is valuable in that it reveals the time scales at which qualitative changes in the solution occur. Similar approaches have also been taken when the thermoviscous diffusion is supplemented by relaxation effects associated with polyatomic molecules, revealing intricate variation in the finest shock scale [6,15]. However, up to this point the asymptotic predictions have never been fully validated.

The ultimate aim of any study of propagation of sonic booms through the atmosphere, with given meteorological data, is to predict shock properties such as amplitude and rise-time at the ground and identify what physical properties control these quantities. As previously noted, direct numerical solutions can be computationally expensive, but more importantly do not necessarily identify the key physical parameter regimes. For this reason, asymptotic analysis undoubtedly has a role to play in understanding the propagation of sonic booms through real atmospheres. However, first a rigorous validation of asymptotic results for simpler systems is required, and this is the primary motivation for the present work.

When studying finite amplitude plane acoustic waves propagating through a thermoviscous medium, Mendousse [1] introduced Burgers' equation in the form

$$\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial \theta} = \nu\frac{\partial^2 v}{\partial \theta^2}, \qquad v(x_0,\theta) = f(\theta)$$

where ν is the viscosity and v represents either particle displacement velocity or perturbation pressure, θ is a retarded time and x is propagation distance. An alternative form,

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x'} = \nu\frac{\partial^2 v}{\partial x'^2}, \qquad v(x',t_0) = \phi(x') \tag{1}$$

is more appropriate for the corresponding initial value problem, with x' the spatial coordinate in the frame moving with the linear sound speed. In each of these forms, quadratic nonlinearity and thermoviscous diffusion act to modify a disturbance travelling at the linear sound speed. By means of a nonlinear transformation [16,17], Burgers' equation can be transformed to the heat equation and hence solved for arbitrary initial conditions. This solution is commonly referred to as the Cole–Hopf solution and, for the plane Burgers' equation (1), can be written as [18]

$$v(x',t) = \frac{1}{t}\frac{\int_{-\infty}^{\infty}(x'-X)\psi(x',X,t)dX}{\int_{-\infty}^{\infty}\psi(x',X,t)dX},$$

where

$$\psi(x',X,t) = \exp\left[-\frac{1}{2\nu}\left(\frac{(x'-X)^2}{2t} + \int_0^X \phi(s)ds\right)\right].$$

The fact that an exact solution exists means that the plane Burgers' equation has been much studied as a test case for numerical methods of solving nonlinear wave equations [19,20].

Subsequently, the formal validity of Burgers' equation as a model for nonlinear acoustics was investigated [2,21]. Extending to a right/outward travelling sound wave which has planar, cylindrical or spherical symmetry [2] and using the notation that all starred variables are dimensional, the equation governing the perturbation velocity of the medium is given by

$$\frac{\partial u^*}{\partial t^*} + \frac{\gamma+1}{2}u^*\frac{\partial u^*}{\partial X^*} + j\frac{u^*}{2t^*} = \frac{1}{2}\Delta\frac{\partial^2 u^*}{\partial X^{*2}}, \qquad X^* = r^* - a_0t^*. \tag{2}$$

Here γ is the adiabatic exponent, a_0 is the small-signal sound speed, r^* is the radial propagation distance, Δ is the diffusivity of sound and $j = 0, 1, 2$ for plane, cylindrical and spherical waves respectively. For a disturbance of typical wavelength l^*, this equation can be derived formally using a multiple scales argument [21], based on the assumptions that:

(i) finite amplitude effects are locally small ($u^*/a_0 \ll 1$);
(ii) geometric spreading effects are small ($l^*/r^* \ll 1$);
(iii) thermoviscous diffusive effects are small ($\Delta/a_0l^* \ll 1$).

This paper is concerned with the physically important case when thermoviscous effects are smaller than nonlinear effects over most of the waveform. However, in order to analyse the structure of the solution in the three cases of planar, cylindrical and spherical spreading it proves most convenient to apply the transformation described in Crighton and Scott [14] (equations (2.3–2.5)). This converts the geometric spreading term into a range dependent viscosity term, leading to the generalised Burgers' equation (GBE),

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} = \epsilon G(T)\frac{\partial^2 U}{\partial X^2}, \tag{3}$$