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Dispersion relation in the limit of high frequency for a hyperbolic system with multiple eigenvalues

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HIGHLIGHTS

- Hyperbolic systems with multiple eigenvalues are studied.
- Plane-wave solutions to linearized hyperbolic systems are considered.
- The dispersion relation in the high-frequency limit is analyzed.
- Recurrence equations for determining the dispersion relation are derived.
- It is shown that linear stability implies stability of weak-discontinuity waves.

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ABSTRACT

The results of a previous paper (Muracchini et al., 1992) are generalized by considering a hyperbolic system in one space dimension with multiple eigenvalues. The dispersion relation for linear plane waves in the high-frequency limit is analyzed and the recurrence formulas for the phase velocity and the attenuation factor are derived in terms of the coefficients of a formal series expansion in powers of the reciprocal of frequency. In the case of multiple eigenvalues, it is also verified that linear stability implies λ-stability for the waves of weak discontinuity. Moreover, for the linearized system, the relationship between entropy and stability is studied. When the nonzero eigenvalue is simple, the results of the paper mentioned above are recovered. In order to illustrate the procedure, an example of the linear hyperbolic system is presented in which, depending on the values of parameters, the multiplicity of nonzero eigenvalues is either one or two. This example describes the dynamics of a mixture of two interacting phonon gases.

1. Introduction

Muracchini, Ruggeri, and Seccia in [1] considered a generic quasi-linear first-order system of partial differential equations in one space dimension:

$$\partial_t \boldsymbol{u} + \boldsymbol{A}(\boldsymbol{u})\, \partial_x \boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}), \quad \boldsymbol{u} = \boldsymbol{u}(t, x), \tag{1.1}$$

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where (∂_t, ∂_x) denotes differentiation with respect to (t, x), $\boldsymbol{u}$ is the $\mathbb{R}^n$ unknown column vector, $\boldsymbol{A}$ is the $n \times n$ matrix, and $\boldsymbol{f}$ is the production term. The system is assumed to be hyperbolic in the time direction, i.e., the eigenvalue problem

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{d} = 0 \tag{1.2}$$

admits real eigenvalues λ and a set of linearly independent right eigenvectors $\boldsymbol{d}$. In (1.2), we use the symbol $\boldsymbol{I}$ to represent the unit $n \times n$ matrix.

The above authors proved that the weak discontinuity waves (the acceleration waves in the language of continuum mechanics) propagating into the region of constant state exist for all times and that the amplitudes of these waves decay under some conditions and conditional initial data (λ-stability).

Moreover, assuming that $\boldsymbol{u}_0$ is a constant equilibrium state, $\boldsymbol{f}(\boldsymbol{u}_0) = \boldsymbol{0}$, these authors linearized system (1.1) in a neighborhood of $\boldsymbol{u}_0$,

$$\partial_t \tilde{\boldsymbol{u}} + \boldsymbol{A}_0\, \partial_x \tilde{\boldsymbol{u}} = \boldsymbol{B}\, \tilde{\boldsymbol{u}}, \quad \boldsymbol{A}_0 := \boldsymbol{A}(\boldsymbol{u}_0),\ \boldsymbol{B} := (\nabla \boldsymbol{f})_0\,, \tag{1.3}$$

and they subsequently considered linear plane waves for the perturbation field $\tilde{\boldsymbol{u}}$,

$$\boldsymbol{u} = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}, \qquad \tilde{\boldsymbol{u}} = \boldsymbol{w} \exp\left[\mathrm{i}\,(\omega\, t - k\, x)\right], \tag{1.4}$$

where ω, the angular frequency, is real and positive and k denotes the complex wave number. They also derived the equation for $\boldsymbol{w}$ (we now omit the index 0 in $\boldsymbol{A}$ and later the tilde in $\boldsymbol{u}$),

$$\left(\boldsymbol{I} - z\boldsymbol{A} + \frac{\mathrm{i}}{\omega}\boldsymbol{B}\right) \boldsymbol{w} = \boldsymbol{0}, \tag{1.5}$$

and the dispersion relation resulting from it:

$$\det\left(\boldsymbol{I} - z\boldsymbol{A} + \frac{\mathrm{i}}{\omega}\boldsymbol{B}\right) = 0, \tag{1.6}$$

where $z := k/\omega$.

Through a formal expansion of the phase velocity $v_{ph} = \omega/\mathrm{Re}(k)$ and of the attenuation factor $\alpha = -\mathrm{Im}(k)$ as a series in powers of $1/\omega$, they proved that in the limit $\omega \to \infty$, the phase velocity approaches a characteristic eigenvalue λ and the attenuation of the plane harmonic wave coincides with that of the weak discontinuity wave. Due to this fact, the linear stability is equivalent to the λ-stability. Furthermore, they derived the recurrence formulas such that it is possible to determine all the coefficients of the power series expansion and such that there is the possibility to obtain a closed form of the dispersion relation by recurrence of the phase velocity $v_{ph} = v_{ph}(\omega)$ and of the attenuation factor $\alpha = \alpha(\omega)$.

We have found that the results given in [1] apply only to those harmonic modes which correspond to simple nonzero eigenvalues. The objective of this paper is to generalize the previous analysis to the case of the harmonic modes corresponding to multiple nonzero eigenvalues. Consequently, for the linearized hyperbolic system (1.3) with multiple nonzero eigenvalues, the propagation of the plane harmonic waves (1.4) is considered and the dispersion relations in forms (1.5) and (1.6) are analyzed. Similarly as in [1], the expansion of the phase velocity and of the attenuation factor in powers of $1/\omega$ is employed. For harmonic modes corresponding to a multiple nonzero eigenvalue, the high-frequency limits of the phase velocity and of the attenuation factor are determined and the recurrence formulas for the expansion coefficients are derived. It is shown that if the nonzero eigenvalue is of multiplicity one, then these formulas reduce to the recurrence equations previously given in [1].

Assuming that the multiplicity of the considered eigenvalue is arbitrary, the conditions for stability of the waves of weak discontinuity are formulated and the condition under which the equivalence of the linear stability and the λ-stability holds is specified.

For system (1.3), the relationship between entropy and stability is also discussed (see Section 4).

To illustrate the procedure and to compare the case of the system having simple nonzero eigenvalues with the case of the system having multiple nonzero eigenvalues, we analyze an example of the linear hyperbolic system of six equations with the right-hand side of relaxation type. In [2, Appendix A], this system has been proposed for the description of the dynamics of a phonon system understood as a mixture of two interacting gases, one of them corresponding to the longitudinal phonons and the second one corresponding to the transverse phonons. If the velocity of longitudinal phonons is assumed to be greater than the velocity of transverse phonons, this system has only simple nonzero eigenvalues. In [3, Section IV D], the simplifying assumption that those velocities are equal has been introduced. As a consequence, the simplified system has all nonzero eigenvalues of multiplicity two.

2. Dispersion relation and its high-frequency limit

2.1. Expansion in powers of the reciprocal of frequency

Let λ_α, $\boldsymbol{l}_\alpha$, and $\boldsymbol{d}_\alpha$ be respectively the eigenvalues, the left eigenvectors, and the right eigenvectors of the linearized system (1.3):

$$\boldsymbol{l}_\alpha\, (\boldsymbol{A} - \lambda_\alpha \boldsymbol{I}) = \boldsymbol{0}, \qquad (\boldsymbol{A} - \lambda_\alpha \boldsymbol{I})\, \boldsymbol{d}_\alpha = \boldsymbol{0}, \quad \alpha = 1, 2, \ldots, n.$$