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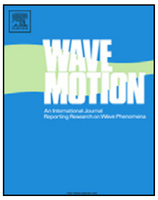

Wave propagation in a fractional viscoelastic Andrade medium: Diffusive approximation and numerical modeling

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HIGHLIGHTS

- The prototypical fractional viscoelastic Andrade model with power-law attenuation is considered.
- The featured fractional derivative is recast using a diffusive representation to reduce memory cost.
- An accurate quadrature scheme is analyzed to compute a stable diffusive approximation.
- The model is implemented in an efficient numerical scheme to perform transient wave simulations.
- Numerical results and a semi-analytical solution demonstrate the performances of the approach.

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ABSTRACT

This study focuses on the numerical modeling of wave propagation in fractionally-dissipative media. These viscoelastic models are such that the attenuation is frequency-dependent and follows a power law with non-integer exponent within certain frequency regimes. As a prototypical example, the Andrade model is chosen for its simplicity and its satisfactory fits of experimental flow laws in rocks and metals. The corresponding constitutive equation features a fractional derivative in time, a non-local-in-time term that can be expressed as a convolution product whose direct implementation bears substantial memory cost. To circumvent this limitation, a diffusive representation approach is deployed, replacing the convolution product by an integral of a function satisfying a local time-domain ordinary differential equation. An associated quadrature formula yields a local-in-time system of partial differential equations, which is then proven to be well-posed. The properties of the resulting model are also compared to those of the Andrade model. The quadrature scheme associated with the diffusive approximation, and constructed either from a classical polynomial approach or from a constrained optimization method, is investigated. Finally, the benefits of using the latter approach are highlighted as it allows to minimize the discrepancy with the original model. Wave propagation simulations in homogeneous domains are performed within a split formulation framework that yields an optimal stability condition and which features a joint fourth-order time-marching scheme coupled with an exact integration step. A set of numerical experiments is presented to assess the overall approach. Therefore, in this study, the diffusive approximation is demonstrated to provide an efficient framework for the theoretical and numerical investigations of the wave propagation problem associated with the fractional viscoelastic medium considered.

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1. Introduction

There is a long history of studies discussing or providing experimental evidences of frequency-dependent viscoelastic attenuations, as observed in e.g. metals [1], acoustic media [2,3] and in the Earth [4,5]. Such a behavior is classically modeled using a fractional derivative operator [6,7], a mathematical tool generalizing to real parameters the standard derivatives of integer orders [8]. While fractional calculus is now a mature theory in the field of viscoelasticity [9], some issues remain commonly encountered. They mostly revolve around the two questions of:

(i) Incorporating fractional dissipation into viscoelastic models that both fit experimental data and have a theoretical validity regarding, e.g., causality properties [10,11] or the Kramers–Kronig relations [12].
(ii) Implementing numerically these fractional models to perform wave propagation simulations. This latter problem is commonly tackled using standard approaches [13] for modeling constant-law of attenuation over a frequency-band of interest, i.e. with the fractional viscoelastic model being approximated by multiple relaxation mechanisms [14].

Bearing in mind the issue (i) discussed above, it is chosen to anchor the present study to a specific, yet prototypical, physically-based viscoelastic model, namely the Andrade model. Initially introduced in [1] to fit experimental flow laws in metals, it has been further investigated in [15]. It is now used as a reference in a number of studies [16–19] for the description of observed frequency-dependent damping behaviors in the field of geophysics and experimental rock mechanics. Moreover, the Andrade model creep function, as written, can notably be decomposed as the sum of a fractional power-law and a standard Maxwell creep function, therefore corresponding rheologically to a spring-pot element arranged in series with a spring-dashpot Maxwell model. Therefore, while being physically motivated and rooted in experiments, this model gives leeway to cover the spectrum from a conventional rheological mechanism to a more complex fractional model, and this with only a few parameters.

This study focuses on the issue (ii), namely the numerical modeling of wave propagation within an Andrade medium that exhibits fractional attenuation. The objective is to develop an efficient approximation strategy of the fractional term featured in this viscoelastic model in view of the investigation and simulation of its transient dynamical behavior. A model-based approach is explored in the sense that one aims at a direct approximation of the original constitutive equation. Therefore, the latter is not intended to be superseded by another viscoelastic model that would be designed to fit only a given observable. For example, the usual approach that employs a multi-Zener model typically approximates the quality factor only.

The article aim and contribution are twofold:

(i) Deploy an approximation of the fractional derivative featured in the constitutive equation considered. A direct discretization of this term, that is associated with a non-local time-domain convolution product [8] requires the storage of the entire variables history, which is out of reach for realistic simulations. The Grünwald–Letnikov approximation of fractional derivatives constitutes a tractable approach, commonly used in viscoelasticity [20]. Its main drawback concerns the stability analysis to be performed for the numerical scheme so-obtained. Indeed, Von-Neumann stability of multistep schemes requires to bound the characteristic roots of the amplification matrix, which may be a difficult task. We do not follow this approach here. Alternatively, a so-called diffusive representation is preferred [21], as it allows to recast the equations considered into a local-in-time system while introducing only a limited number of additional memory variables in its discretized form [22]. Following later improvements of the method in [23–26], an efficient quadrature scheme is investigated in order to obtain a satisfactory fit of the reference model compliance.
(ii) Implement the resulting approximated model into a wave propagation scheme. While the available literature on the numerical simulation of transient wave propagation within fractionally-damped media is relatively scarce, see e.g. [27,28], the aim is here to demonstrate the efficiency of the proposed approach. For the sake of simplicity, the viscoelastic medium considered is assumed to be unidimensional and homogeneous. After discretization of the dynamical system at hand, a Strang splitting approach [29] is adopted, both to reach an optimal stability condition and to enable the use of an efficient time-marching scheme coupled with an exact integration step. Moreover, deriving a semi-analytical solution for the configuration considered, as a baseline, a set of numerical results is presented to assess the quality of the numerical scheme developed. The overall features and performances of the diffusive representation are finally discussed to compare the Andrade model with its diffusive approximated counterpart.

Notably, this study demonstrates that the behavior of fractional viscoelastic models such as the Andrade model can be correctly described using a diffusive approximation. The resulting model is shown to be well characterized mathematically while being easily tractable numerically in view of performing simulations in the time domain.

This article is organized as follows. The Andrade model is presented and discussed in Section 2. Considering the featured fractional derivative, a corresponding diffusive approximated (DA) version of the former is subsequently formulated and referred to as the Andrade-DA model. The evolution problem is investigated in Section 3, with the derivation and analysis of the first-order hyperbolic system associated with the Andrade-DA model. Section 4.1 is concerned with the definition and computation of an efficient quadrature scheme for the diffusive approximation, while the implementation of the fully discretized system is described in Section 4.2. Corresponding numerical results are presented and discussed in Section 5.