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SOME SPECIAL FUNCTIONS IDENTITIES ARISING FROM COMMUTING OPERATORS

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Commuting is an important property in many cases of investigation of properties of operators as well as in various applications, especially in quantum physics. Using the observation that the generalized weighted differential operator of order k and the weighted Hardy-type operator commute we derive a number of new and interesting identities involving some functions of mathematical physics.

Keywords: commuting operators, Hardy-type operator, special functions, integral and differential identities.

1. Introduction

When studying two operators P, Q of quantum theory it is crucial whether the relation

$$[P, Q] = 0, \quad \text{where} \quad [P, Q] = (-\imath\hbar)^{-1}(PQ - QP),$$

is fulfilled or not. Such physical quantities, for which there are uniquely assigned operators P, Q, are simultaneously measurable if and only if $[P, Q] = 0$. Equivalently, we say that the *operators* P, Q *commute* if the commutator $[P, Q] = 0$, i.e. $PQ = QP$. There are many known operators for which the commutation relation is fulfilled, e.g. the class of normal operators. On the other side, non-commutative operators are source of some interesting stories in mathematics and physics, e.g. the famous Heisenberg uncertainty principle saying that it is impossible to know the momentum and position of a particle simultaneously (see [6]).

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In this paper we are interested in two operators, namely the *Hardy-type operator* $\mathbf{H}_w$ defined by

$$\mathbf{H}_w f(x) = \frac{1}{w_1(x)} \int_\alpha^x w(t) f(t)\, dt, \qquad x > 0,$$

where $-\infty \leq \alpha \leq \infty$, $w_1(x) = \int_\alpha^x w(t)\, dt$ and w, f are real measurable, locally integrable functions, and the *generalized weighted differential operator* $\mathbf{D}_k^w$ of order k given by

$$\mathbf{D}_k^w f(x) = \left(\frac{w_1(x)}{w(x)} \frac{d}{dx}\right)^k f(x) = \left(\frac{w_1(x)}{w(x)} \frac{d}{dx}\right)^{k-1} \left(\frac{w_1(x)}{w(x)} f'(x)\right),$$

$k = 1, 2, \ldots$. Note that operators $\mathbf{H}_w$ are of interest in various functional spaces mainly in connection with Hardy inequality, cf. [4]. The case $\alpha = 0$ and $w \equiv \text{const}$ corresponds to the Hardy's averaging operator (or, Hardy's arithmetic mean operator), therefore we propose to call $\mathbf{H}_w$ the Hardy-type operator. On the other hand, operators $\mathbf{D}_k^w$ appear in the theory of differential equations (see e.g. [5] and [7]). If $w_1(x)/w(x) = 1$, the resulting k-th order differential operator will be simply denoted by

$$\mathbf{D}_k f(x) = \frac{d^k}{dx^k} f(x), \qquad k = 1, 2, \ldots,$$

as usual. A generalization of weighted Hardy's averaging operator is provided in [3], where the general mean-type inequality involving such operator is investigated.

The first result of this paper is the observation that the operators $\mathbf{D}_k^w$ and $\mathbf{H}_w$ commute for each $k \in \mathbb{N}$. Using this fact in Section 3 we establish a few interesting new identities involving some special functions of mathematical physics.

2. A note on commutation relation

Now we describe an easy observation on commutativity of operators $\mathbf{D}_k^w$ and $\mathbf{H}_w$. For the sake of brevity let us replace $\mathbf{H}_w h(x)$ by $H(x)$. Then we have

$$H(x) = \frac{1}{w_1(x)} \int_\alpha^x w(t) h(t)\, dt. \tag{1}$$

Differentiating the equality (1) and then using integration by parts with $w_1(\alpha) = 0$ we obtain the identity

$$H'(x) = \frac{w(x) \int_\alpha^x w_1(t) h'(t)\, dt}{w_1(x)^2},$$

which may be written as follows,

$$\begin{aligned} \frac{w_1(x)}{w(x)} H'(x) &= \frac{1}{w_1(x)} \int_\alpha^x w_1(t) h'(t)\, dt \\ &= \frac{1}{w_1(x)} \int_\alpha^x w(t) \left(\frac{w_1(t)}{w(t)} h'(t)\right) dt, \end{aligned} \tag{2}$$