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Self-adapting absorbing boundary conditions for the wave equation

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ABSTRACT

In this paper, we introduce a self-adapting absorbing boundary condition for the linear wave equation. The construction is based on a local computation of the incidence angle of the outgoing wave and on the use of the classical lowest order Engquist–Majda absorbing boundary condition. In order to obtain a good approximation of the incidence angle, we decompose adaptively the absorbing boundary into subsegments and apply locally the Fourier transformation. Numerical results illustrate the performance of the newly designed self-adapting absorbing boundary condition and show its robustness.

1. Introduction

Over the past thirty years, absorbing boundary conditions (ABCs) have been developed into a vigorous research direction including a wide spectrum of methods and approaches. The description of these techniques is out of the scope of the paper. For a detailed discussion on the topic, we refer the reader to the comprehensive review articles [1–5] and the references therein. Despite a huge amount of effort applied to the investigation of transparent boundary conditions and a number of developed methods, only few of them are used in real-life applications. Usually the choice of ABCs is dictated by the compromise between the efficiency of the ABCs and the complexity of the implementation. The method proposed in this paper allows to adapt the ABCs to the angle at which the outgoing wave is impinged on the absorbing boundary. It is easy to implement and the additional computational cost is small. A similar idea was exploited in [6,7] for the Schrödinger equation both in one and two dimensions while its application to the two dimensional wave equation is studied in this work for the first time. Moreover, the main difference of the method proposed in the current work is that it automatically divides the boundary into segments to obtain better absorption rates while in [7] this parameter is kept constant. Besides, the adaptation method proposed in [6] may lead to instabilities [8]. In the present work, no instabilities are observed.

Two different approaches based on self-adaptation of the ABCs to the incidence angle of the wave can be realized. The first one is rather naive and based on the adjustment of the absorbing boundary itself rather than the ABCs. In other words, one can derive the ABCs for a preassigned angle of the outgoing wave. By knowing this angle, the absorbing boundary can be tuned to have the prescribed angle between the boundary and the direction of the wave propagation. This approach requires a reconstruction of the grid that may result in a severe performance degradation and it hardly finds any application in real-life problems. Alternatively, the boundary conditions can be adapted to the incidence angle of the outgoing wave on the fly. In this case, one has to detect the direction of the wave propagation and tune the ABCs to the angle between the boundary and the direction in which the wave travels. This method is proposed in the paper.

In this work, we focus on local, both in space and time, absorbing boundary conditions proposed by Engquist and Majda in [9]. We introduce the problem statement, review the first and second order Engquist–Majda ABCs, and derive their

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analogue for an arbitrary angle of incidence in Section 2. The proposed method is described in Section 3. Section 4 is devoted to the discretization of the continuous problem. In Section 5, numerical experiments demonstrating the accuracy of the method are presented. Finally, we discuss the proposed approach in Section 6.

2. Governing equations and absorbing boundary conditions

The propagation of acoustic waves in a bounded two-dimensional domain $\Omega \subset \mathbb{R}^2$ can be described by the scalar wave equation

$$c^{-2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. \tag{1}$$

Here c is the speed of sound which is supposed to be constant in our considerations and u is a function of time t and Cartesian coordinates (x, y). The wave equation (1) is equipped with an inhomogeneous Dirichlet boundary condition

$$u(x, y, t) = g(t) \quad \text{on } \Gamma_T \times (0, t_n]$$

and suitable ABCs on the absorbing boundary $\Gamma_{ABC} \times (0, t_n]$. Here $\Gamma_{ABC} = \partial\Omega \setminus \Gamma_T$ and Γ_T is assumed to have a positive measure. The initial conditions for the wave equation (1) are

$$u(x, y, 0) = u_0(x, y), \qquad \frac{\partial}{\partial t}u(x, y, 0) = u_1(x, y) \quad \text{in } \Omega.$$

For the convenience of the reader, we briefly review the first and second order Engquist–Majda ABCs first derived in [9]. The derivation of these transparent boundary conditions is based on the Fourier transformation of the wave equation (1) in the (y, t) variables. Denoting the Fourier transformation of $u(x, y, t)$ in the variables (y, t) by $\hat{u}(x, \eta, \omega)$, we get

$$L = c^{-2}(i\omega)^2 - \left(\frac{\partial^2}{\partial x^2} + (i\eta)^2\right), \qquad L\hat{u} = 0.$$

The operator L can be factorized as follows

$$L = -\left(\frac{\partial}{\partial x} - \sqrt{c^{-2}(i\omega)^2 - (i\eta)^2}\right)\left(\frac{\partial}{\partial x} + \sqrt{c^{-2}(i\omega)^2 - (i\eta)^2}\right). \tag{2}$$

One of the factors in (2) can be used to define suitable transparent boundary conditions. For instance, the boundary condition

$$\left(\frac{\partial}{\partial x} - \sqrt{c^{-2}(i\omega)^2 - (i\eta)^2}\right)\hat{u} = 0 \tag{3}$$

annihilates outgoing waves on the boundary $x = 0$. Although the boundary condition (3) is quite attractive from the theoretical point of view, it is rarely used in numerical simulations. The non-local character of this boundary condition makes it computationally expensive. Replacing the square root in the boundary condition (3) by a Taylor series is a starting point for defining local ABCs of a given order. The first order and the second order Engquist–Majda ABCs can then be expressed as

$$\frac{\partial u}{\partial \mathbf{n}} - c^{-1}\frac{\partial u}{\partial t} = 0 \quad \text{at } \Gamma_{ABC} \times (0, t_n] \tag{4a}$$

and

$$c^{-1}\frac{\partial^2 u}{\partial \mathbf{n}\partial t} - c^{-2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{2}\frac{\partial^2 u}{\partial \boldsymbol{\tau}^2} = 0 \quad \text{at } \Gamma_{ABC} \times (0, t_n], \tag{4b}$$

respectively. Here, $\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit normal and tangential vectors to the absorbing boundary Γ_{ABC}.

The Engquist–Majda ABCs (4) were constructed for normal incidence that provides a perfect absorption of the wave hitting the boundary at this specific angle which is denoted by θ. Consequently, the deviation from a zero angle results in higher reflections. To get a better feeling for the influence of angle θ, we consider the incident

$$u_I(x, y, t) = e^{i(\omega t - \zeta x + \eta y)} \tag{5a}$$

and the reflected

$$u_R(x, y, t) = Ae^{i(\omega t + \zeta x + \eta y)}, \tag{5b}$$

waves, where A is the amplitude of the reflected wave. The waves (5a)–(5b) fulfill the wave equation (1) for any $\zeta, \eta, \omega \in \mathbb{R}$ that satisfy the dispersion relation

$$\omega^2 = \zeta^2 + \eta^2.$$

Our goal is to determine the amplitude A of the reflected wave u_R. To this end, we substitute the sum of the incident and the reflected waves into the boundary condition (4a) which provides

$$A_1 = \frac{\cos(\theta) - 1}{\cos(\theta) + 1} \tag{6a}$$