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Ward identities for visco-acoustic and visco-elastic propagation media

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ABSTRACT

This study discusses Ward identities in the presence of viscous dissipation. A Ward identity relates the Green function of the medium to the noise correlation function. Our study is focused on two types of mechanical waves: the scalar (1-component) pressure field, and the 3-component displacement field. Under some realistic (from a practical point of view) low attenuation and far-field assumptions, the first-order time-derivative of the noise correlation is shown not to be proportional to the odd part of the Green function any longer. New algebraic relations are derived in the Fourier domain, and a new form of the Ward identity is obtained that relates the third-order time-derivative of the noise correlation function to the odd part of the Green function.

1. Introduction

Passive identification of a propagation medium consists of retrieving the medium parameters using uncontrolled noise fluctuations. This idea has long been pursued in acoustics [1,2] and seismology [3,4], and it has given rise to numerous applications and experimental validations [5–7].

Previous studies [1,3,5–7] have addressed the problem by introducing the estimation of the Green function of the medium. This estimation is made possible by exploiting a Ward identity [8], which relates the noise correlation function to the Green function [1,2,9]. The fundamental role of dissipation in the Ward identity has been outlined previously [7–9], where a constant dissipation model has been assumed. Such simplification is not realistic from a physical point of view [10,11], and needs to be discussed further.

In this study, a more realistic dissipation model that accounts for viscosity is introduced into the derivation of the Ward identity. The proposed derivation for visco-acoustic waves (*i.e.* the study of the pressure field) is presented in Section 2. Section 3 deals with visco-elastic waves (*i.e.* the study of the displacement field). The Ward identities derived for viscous damping are difficult to use in practice, as this involves the inverse of the dissipation operator, which generally cannot be easily calculated. To overcome this difficulty, approximated Ward identities are derived in the framework of a low dissipation and far-field approximation.[1] Our results are valid for unbounded,[2] homogeneous and isotropic elastic media.

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[1] This second assumption only concerns the displacement field.

[2] Unbounded property is realistic when the wavelength of a propagated wave is negligible with respect to the dimensions of the media. Note that this is fully compatible with the elastic case far-field assumption introduced above.

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Here, the noise correlation function will be referred to as the Green correlation,[3] as it is defined as the cross-correlation of a propagated white noise source. Thus the Green correlation can be seen as the Green function "of the order 2", and it depends solely on the medium properties.

2. The visco-acoustic framework

In this section, visco-acoustic propagation media are considered. First, the propagation law satisfied by the pressure field in such media is reconsidered, and both the Green function and the Green correlation are introduced. Then, an exact Ward identity is proposed, elaborated from the pioneering work of [8]. A low viscous damping approximation is developed, which turns out to make it much easier to interpret the formulas for the Ward identities.

2.1. Visco-acoustic wave equations

We consider a *linear* homogeneous isotropic and unbounded visco-acoustic propagation medium. Let v denote the speed of sound, and α^2 the viscous coefficient. Let $\mathbf{p}$ be the pressure field generated by a source $\mathbf{f}$ exciting the medium. The propagation equation is thus [11]:

$$\left[\frac{\partial^2}{\partial t^2} - \alpha^2 \Delta \frac{\partial}{\partial t} - v^2 \Delta\right] \mathbf{p} = \mathbf{f}. \tag{1}$$

Note that for solving Eq. (1), both the initial conditions (causality of the pressure field and its time-derivative) and the boundary conditions (all of the fields vanish at infinity) need to be specified.

Thus, from Eq. (1), we can deduce the visco-acoustic dispersion law:

$$k^2(\omega) = \frac{\omega^2}{\mathbf{v}^2(\omega)} \tag{2}$$

where $\mathbf{v}(\omega)$ is the complex propagation speed of visco-acoustic waves:

$$\mathbf{v}^2(\omega) = v^2 + \mathbf{i}\omega\alpha^2. \tag{3}$$

For low viscous damping, this becomes:

$$k(\omega) \approx \frac{\omega}{v}. \tag{4}$$

This approximation will be useful to derive the approximate Ward identities.

2.2. Visco-acoustic Green function and Green correlation

The visco-acoustic Green function $\mathbf{G}$ relates the source $\mathbf{f}$ to the generated pressure field $\mathbf{p}$ according to:

$$\mathbf{p} = \mathbf{G} \otimes_T \otimes_S \mathbf{f} \tag{5}$$

where $\otimes_T$ and $\otimes_S$ are the time and space convolution, respectively.

The principle of passive identification relies on recording acoustic signals at two locations and using their statistical properties to retrieve medium parameters when source fields are not controlled. A consequence of Eq. (5) is that the Green function is nothing but the pressure field obtained for a point source emitting an impulse signal. Otherwise stated, $\mathbf{G}$ is the impulse response of the medium. Similarly, the visco-acoustic Green correlation $\mathbf{C}$ is the cross-correlation of a field that is generated by a white (in time and space) noise source. Its expression can be expressed as [8]:

$$\mathbf{C} = \mathbf{G} \otimes_T \otimes_S \mathbf{G}^- \tag{6}$$

where $\mathbf{G}^-$ represents the time-reversed Green function.

2.3. Visco-acoustic Ward identity

Elaborating on Weaver's work, an exact Ward identity relating the Green function to the Green correlation in the visco-acoustic case can be derived as[4]:

$$\frac{\partial \mathbf{C}}{\partial t} = \alpha^{-2} \Delta^{-1} \mathrm{Odd}\mathbf{G} \tag{7}$$

[3] This terminology was introduced in [12].

[4] Reformulated with the notation used in the present study, Weaver's result can be written as: $\mathbf{G} \otimes_T \otimes_S \left[\alpha^2 \frac{\partial}{\partial t} \Delta \mathbf{G}^-\right] = \mathrm{Odd}\mathbf{G}$.