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# Space–time transformation acoustics

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## HIGHLIGHTS

- We derive the mappings under which the acoustic wave equations are form invariant.
- We thoroughly compare Standard (STA) and Analogue (ATA) Transformation Acoustics.
- We show that the pressure wave equation is not suited for an ATA approach.
- We design an acoustic frequency converter via ATA that cannot be obtained with STA.

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## ABSTRACT

A recently proposed analogue transformation method has allowed the extension of transformation acoustics to general space–time transformations. We analyze here in detail the differences between this new analogue transformation acoustics (ATA) method and the standard one (STA). We show explicitly that STA is not suitable for transformations that mix space and time. ATA takes as starting point the acoustic equation for the velocity potential, instead of that for the pressure as in STA. This velocity-potential equation by itself already allows for some transformations mixing space and time, but not all of them. We explicitly obtain the entire set of transformations that leave its form invariant. It is for the rest of transformations that ATA shows its true potential, allowing for building a transformation acoustics method that enables the full range of space–time transformations. We provide an example of an important transformation which cannot be achieved with STA. Using this transformation, we design and simulate an acoustic frequency converter via the ATA approach. Furthermore, in those cases in which one can apply both the STA and ATA approaches, we study the different transformational properties of the corresponding physical quantities.



## 1. Introduction

The success of transformation optics [1–4], together with the availability of artificial materials with tailor-made properties [5,6], has led researchers to explore the possibility of applying similar techniques in other branches of physics. Outside of

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optics, acoustics is probably the field in which the greatest advance has been achieved. The form-invariance of the acoustic equations under spatial transformations is used to obtain the material parameters that deform acoustic space in the desired way. One of the most important applications of this technique is the cloaking of acoustic waves [7–12].

One possible problem in this process is that the transformations from virtual to physical space may result in metamaterials that cannot be realized in practice. In order to overcome this problem, several authors [13,14] have proposed to invert the process by first studying the range of realizable material parameters, and then deriving the appropriate transformations which guarantee the desirable effect, such as acoustic cloaking. Another problem, and the one that we will mainly address here, is related to the transformation process itself. Unlike electromagnetic theory, classical acoustics is based on non-relativistic equations that are non-invariant under transformations that mix space and time. As a consequence, the standard method for transformation acoustics cannot be applied to design devices based on this kind of transformation, contrarily to what has been done in optics [15–17].

Recently, the construction of a general transformation acoustics formalism was tackled in a different way [18,19]. Instead of transforming directly the acoustic equations, the symmetries of an analogue abstract space–time (described by relativistic equations) were exploited. In this method, each couple of solutions connected by a general coordinate transformation in the analogue space–time can be mapped to acoustic space. In this way, it is possible to find the relation between the acoustic material parameters associated with each of these transformation-connected solutions. This method is referred to as analogue transformation acoustics (ATA) and revolves around the acoustic velocity potential wave equation and its formal equivalence with the relativistic equation that describes the evolution of a scalar field in a curved space–time [20,21].

Since ATA and STA start from different initial equations (STA relies on pressure equations, whereas ATA starts from the velocity potential), it is worth studying the differences between the two methods. The first question that arises is whether it could be possible to construct an analogue transformation method based on the pressure wave equation, rather than the velocity potential formulation, and what its range of application would be. Second, it would be desirable to know if the pressure transforms in the same way in STA and ATA in those cases in which both methods can be used. Finally, we would like to explicitly obtain the set of transformations under which the acoustic equations are directly form-invariant in the original acoustic laboratory space–time. In all the transformations that fall outside this set, the construction of the auxiliary relativistic analogue space–time, and hence the use of ATA, is essential to achieve the desired transformation. All these questions are addressed in this work. In addition, to illustrate the potential of ATA, we analyze an example of a non-form-preserving transformation, namely, a space-dependent linear time dilation, which cannot be considered within STA. Using this transformation, we design and numerically test an acoustic frequency converter.

The paper is organized as follows. In Section 2 we outline the main limitation of the approach based on transforming directly the acoustic equations and present the set of transformations that do not preserve the form of the velocity potential equation (the detailed derivation can be found in the Appendix). In Section 3 we first review the ATA method. Then, we explicitly demonstrate that, although an analogue approach based on the pressure wave equation can in principle be constructed, it is not suitable for transformations that mix space and time. In Section 4, we design and analyze the above-mentioned acoustic frequency converter. The differences between STA and ATA are studied in depth in Section 5. Finally, conclusions are drawn in Section 6.

## 2. General space–time transformations

The various existing analyses in STA start from the following basic equation for the pressure perturbations $p$ of a (possibly anisotropic) fluid medium [22]:

$$\ddot{p} = B\nabla_i \left(\rho^{ij}\nabla_j p\right). \tag{1}$$

Here, $B$ is the bulk modulus and $\rho^{ij}$ the (in general, anisotropic) inverse matrix density of the background fluid. We will use latin spatial indices $(i, j)$ and Greek space–time indices ($\mu$, $\nu$, with $x^0 = t$). This is a Newtonian physics equation so that $\nabla$ represents the covariant derivative of the Newtonian flat 3-dimensional space. In generic spatial coordinates it will read

$$\ddot{p} = B\frac{1}{\sqrt{\gamma}}\partial_i \left(\sqrt{\gamma}\rho^{ij}\partial_j p\right), \tag{2}$$

where $\gamma$ is the determinant of the three-dimensional spatial metric $\gamma_{ij}$ (with $\gamma^{ij}$ its inverse). The success of STA relies on the form invariance of this equation under spatial coordinate transformations. It is easy to prove, however, that Eq. (2) is not form invariant for more general (space–time mixing) transformations.

Another commonly used equation in acoustics is the one describing the evolution of the potential function $\phi_1$ for the velocity perturbation $\mathbf{v}_1$ defined as $\mathbf{v}_1 = -\nabla\phi_1$ [20,21][1]:

$$-\partial_t \left(\rho c^{-2}\left(\partial_t\phi_1 + \mathbf{v}\cdot\nabla\phi_1\right)\right) + \nabla\cdot\left(\rho\nabla\phi_1 - \rho c^{-2}\left(\partial_t\phi_1 + \mathbf{v}\cdot\nabla\phi_1\right)\mathbf{v}\right) = 0, \tag{3}$$

[1] Note that this definition does not impose any restriction on the vorticity of the background flow. In fact, even when the fluid is rotational, the present formalism can be maintained for sound waves satisfying $\omega \gg \omega_0$, with $\omega_0$ the rotation frequency of the background fluid and $\omega$ that of the acoustic perturbation. See the discussion in [23].