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The discrete Lamb problem: Elastic lattice waves in a block medium

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HIGHLIGHTS

- An asymptotic behavior of perturbations for the transient Lamb problem for a point step load in a discrete medium is obtained.
- Numerical examples are given to demonstrate the accuracy of our asymptotic solutions.
- A new approximate representation of the Lommel function $s_{0,k}$ for $k \gg 1$ in terms of the Scorer function Gi is obtained.

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ABSTRACT

We study the propagation of transient waves under the action of a vertical step point load on the surface of a half-space filled by a block medium. The block medium is modeled by a square lattice of masses connected by springs in the directions of the axes x, y, and in the diagonal directions. The problem is solved by two methods. Analytically, we obtain asymptotic solutions in the vicinity of the Rayleigh wave at large time intervals. Numerically, we obtain a solution for any finite time interval. We compare these solutions with each other and with the solution to the Lamb problem for an elastic medium.

1. Introduction

In [1], M.A. Sadovskiy has shown that, in the study of rock masses, it is necessary to take into account their block-hierarchical structure. As noted in [1,2], the structure of a block medium is the cause of various dynamic phenomena that are absent in a homogeneous medium and, therefore, cannot be described by its models. Among those dynamic phenomena we distinguish the propagation of the pendulum waves having a low velocity of spread and long length under a pulse action in rock masses with the block structure. Some peculiarities of the pendulum waves were studied on one-dimensional models of the block media in [3,4]. In these papers we calculated the wave motion in a chain of elastic rods, separated by compressible intermediate layers, and showed that a low-frequency disturbance caused by the pulse action is well described in the model of "rigid blocks and viscoelastic layers". The same approach was used in [5,6] for the description of the dynamic behavior of a two-dimensional block medium in which rigid blocks are assumed to be of a rectangular shape. A simplified model of a block medium can be obtained if we treat the blocks as point masses connected by springs. In this case, a block medium

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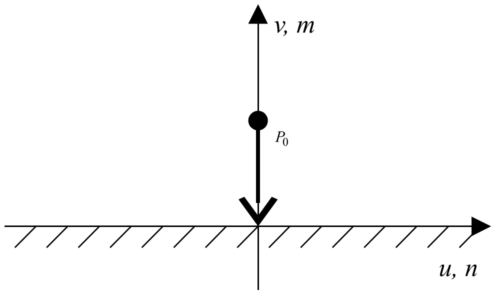

Fig. 1. Setting of the problem.

can be represented as a lattice of masses connected with each other by springs. In [7–10], different versions of this approach were used in the plane and anti-plane settings.

For the first time, the problem of the dynamic impact of a vertical point force applied to the boundary of an elastic half-space was considered by H. Lamb [11]. In that work he obtained integral representations for the displacements on the boundary of the half-space and showed that, on the surface of the half-space, the Rayleigh waves [12] propagate along with the *P*- and *S*-waves. A similar integral representation of a solution to the Lamb problem, both two-dimensional and three-dimensional, was obtained in [13] by the method of functional-invariant solutions. In [14], L. Cagniard obtained a solution to the three-dimensional Lamb problem using the method of integral transformations. To calculate the inverse integral transforms, he proposed a method that now bears his name. Later that method was developed and generalized in [15]. Without going into details, we mention that the Lamb problem and Rayleigh waves were considered in many books, see, e.g., [16–21], and papers, see, e.g., [22,23], devoted to the issues of wave propagation. In most of these books and papers the Cagniard–De Hoop method is used.

The static setting of the plane Lamb problem, when the load does not depend on time, is called the Flamant problem. In this setting the displacement field is given in [24]. A solution for a similar problem for the stress field is given in [18]. In [18] an approximate solution is also obtained for the stationary Lamb problem for displacements on the surface of the half-space under the action of a harmonic point load. The stationary Lamb problem is also considered in [25] within the framework of the Cosserat continuum.

In the transient setting, analytical solutions to the plane Lamb problem for elastic media were considered in [16,17, 19–23]. In [26,27], asymptotic solutions are obtained in the vicinity of the quasi-front for the case of the action of an instantaneous point force on a semi-infinite plate; the method of contour integrals is used and the solution is represented in the integral form.

In the transient setting, numerical solutions to the Lamb problem for elastic media were considered in [28], using the finite-element method for plane and spatial problems for the case of harmonic loading and in [29], using the finite-difference method for the plane problem for the case of distributed loading on the free surface.

For a discrete medium composed of a lattice of point-masses that simulates a continuous elastic medium, the Lamb problem about the propagation of disturbances was solved in [7], where a solution was obtained for 2D- and 3D-lattices in the form of multiple integrals, which are similar to the integral representations of the Bessel functions. Thus, in [7,29], various discretizations of the elastic medium for the Lamb problem were considered. Unlike [7,29], in this paper we obtain both a finite-difference solution and an analytical solution to the Lamb problem for a discrete medium and compare these solutions with each other and with the solution for an elastic medium given in [16,23].

2. Setting of the problem

In the present paper, we study the transient plane Lamb problem on the impact of a vertical point load on the boundary of a half-plane filled by a block medium as it is shown in Fig. 1, where u is the horizontal displacement, v is the vertical displacement, n, m are the indices of the masses in the directions x, y, and P_0 is the amplitude of the point load. The block medium is modeled by a uniform two-dimensional lattice consisting of the masses connected by springs in the directions of the axes x, y, and in the diagonal directions (Fig. 2). This is a special case of the model proposed in [8].

The equations of the motion of the mass with indices n, m, located apart from the boundary, are as follows:

$$\begin{aligned} M\ddot{u}_{n,m} &= k_1(u_{n+1,m} - 2u_{n,m} + u_{n-1,m}) + k_2(u_{n+1,m+1} + u_{n-1,m-1} + u_{n+1,m-1} \\ &\quad + u_{n-1,m+1} - 4u_{n,m})/2 + k_2(v_{n+1,m+1} + v_{n-1,m-1} - v_{n-1,m+1} - v_{n+1,m-1})/2, \\ M\ddot{v}_{n,m} &= k_1(v_{n,m+1} - 2v_{n,m} + v_{n,m-1}) + k_2(u_{n+1,m+1} + u_{n-1,m-1} - u_{n+1,m-1} \\ &\quad - u_{n-1,m+1})/2 + k_2(v_{n+1,m+1} + v_{n-1,m-1} + v_{n-1,m+1} + v_{n+1,m-1} - 4v_{n,m})/2. \end{aligned} \tag{1}$$

Here M is the mass of a block; k_1 is the spring stiffness in the directions of the axes x, y; k_2 is the spring stiffness in the diagonal directions. The initial conditions are supposed to be zero:

$$u_{n,m} = \dot{u}_{n,m} = v_{n,m} = \dot{v}_{n,m} = 0.$$