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Unsteady motions of a new class of elastic solids

K. Kannan [a], K.R. Rajagopal [b,*], G. Saccomandi [c]

[a] Department of Mechanical Engineering, Indian Institute of Technology Madras, Chennai, 600 036, India
[b] Department of Mechanical Engineering, Texas A & M University, College Station, TX 77840, USA
[c] Dipartimento di Ingegneria Industriale, Università di Perugia, 06125 Perugia, Italy

HIGHLIGHTS

- Wave propagation in a new class of elastic bodies is studied.
- The new class of bodies does not belong to Cauchy elastic or Green elastic bodies.
- It is shown that stress waves can develop.

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ABSTRACT

In this short paper we study unsteady motions of a new class of elastic solids, wherein one can justify a non-linear relationship between the linearized strain and the stress, an impossibility within the classical construct of elasticity. For the class of materials concerned, one has to solve simultaneously the balance of mass, balance of linear momentum and the constitutive relation. In general, one has ten scalar unknowns, i.e., density ρ, the components of Cauchy stress $\boldsymbol{T}$ and displacement $\boldsymbol{u}$, and ten scalar algebraic–partial differential equations, the balance of mass (1), the balance of linear momentum (3) and the constitutive equation (6). The stress wave that is generated is quite distinct from what one observes within the context of the classical theory.

1. Introduction

Recently there has been interest in studying a new class of elastic bodies that are described by an implicit relationship between the stress and the stretch (see Rajagopal [1,2], Rajagopal and Srinivasa [3], Rajagopal [4]), and the sub-class wherein the stretch is given in terms of the stress. The class of fully implicit models includes Cauchy and Green elastic bodies (see Truesdell and Noll [5]) as special sub-classes.[1] In the case of classical Cauchy Elastic or Green Elastic bodies, we are inexorably led to the classical linearized elastic model when we make the assumption that the displacement gradient is small. However, in the new class of elastic bodies, the same assumption concerning the displacement gradient can lead to a non-linear relationship between the linearized strain and the stress. The implications of such a possibility cannot be overemphasized.

There is a large amount of data on Gum metal and Titanium alloys, as well as other alloys, which clearly show that in one-dimensional experiments, the relationship between the strain and the stress is discernibly non-linear even for strains less than 0.01 (see the experimental work of Saito et al. [9] and the paper by Rajagopal [10]). Such strains, or to be more precise

* Corresponding author.
E-mail addresses: krishnakannan@iitm.ac.in (K. Kannan), krajagopal@tamu.edu (K.R. Rajagopal), giuseppe.saccomandi@unipg.it (G. Saccomandi).

[1] Recently, Carroll [6] has shown that a Cauchy elastic body that is not Green elastic would exhibit unacceptable physical response characteristics; it would be capable of generating work in a closed cycle and hence become an infinite source of energy (see also Green [7] and Rivlin [8]).

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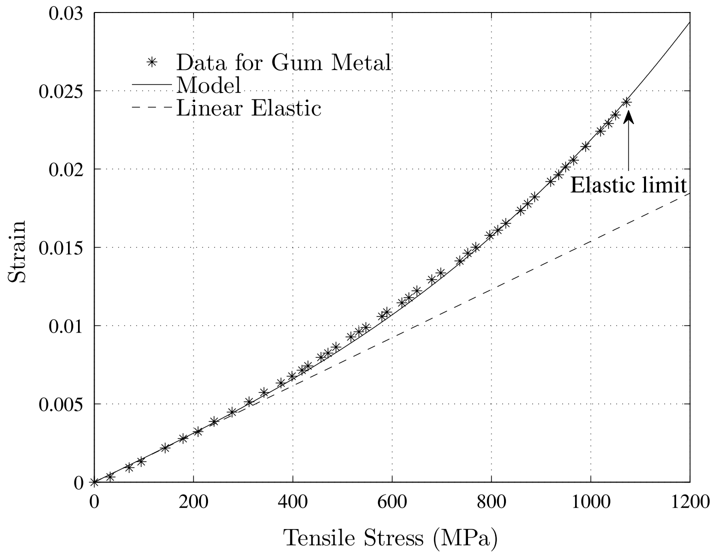

Fig. 1. Cold worked titanium alloy (Gum metal) described by material parameters $n = 0.79$, $\alpha = 1.077\text{e}{-}05\ \text{MPa}^{-1}$, $\beta = 4.615\text{e}{-}06\ \text{MPa}^{-1}$ and $\gamma = 1.631\text{e}{-}06\ \text{MPa}^{-2}$ exhibiting non-linear stress–strain behavior at room temperature.
Source: The figure is replotted using the data reported by [9].

the norm of the displacement gradient that gives rise to such strains are sufficiently small that the square of the norm of the displacement gradient can be neglected in comparison to the value of the norm of the displacement gradient, and hence the Cauchy elastic model would reduce to the model of linearized elasticity, and such a model would be incapable of describing the above mentioned experiments. In Fig. 1, an enlargement of experimental results from a paper by Saito et al. [9] in Science on the response of Gum metal, is provided. Saito et al.'s [9] work has led to numerous experimental studies that confirm their observation (see, for example, [11–14]). The point to take away from all these papers is that there are numerous experiments which present data that unmistakably and indubitably indicate that the strain–stress relationship is non-linear and *the linearized model cannot be used to describe such response*. As mentioned earlier, a class of materials that can describe such a non-linear relationship between the linearized strain and the stress is the class introduced by Rajagopal [1,2]. Here, we study the propagation of waves in such a class of materials. In Fig. 1 we see that the departure from linearity occurs for strains of the order of 0.005, definitely in the range of what would be considered 'small strains'. It is important to recognize, unlike the fully non-linear model, the linearized model that is obtained from the generalization of the class of elastic bodies introduced by Rajagopal [1,2] is only an approximation, and in keeping with the classical linearized theory, the linearized model does not satisfy frame-indifference as the linearized strain is not frame-indifferent.

The fact that one has a non-linear relationship between the linearized strain and the stress makes such elastic bodies suitable to study problems wherein, the strains become unacceptably large within the context of the classical linearized elastic theory. A class of applications which immediately comes to mind concerns the fracture of brittle elastic bodies wherein cracks are initiated without the onset of inelasticity. Within the context of classical linearized elasticity, the strain is of $O(1/\sqrt{r})$ as one approaches the crack tip, r being the radial distance from the tip of the crack. Another problem where one encounters the blowing up of the strain is due to a concentrated load applied on a body. Recently, it has been shown that within the new class of problems wherein one has a non-linear relationship between the linearized strain and the stress one can obtain bounded strain at the tips of cracks (see Rajagopal and Walton [15]) and near the tip of a V-notch (Kulvait et al. [16]), as well as in several other boundary value problems. There has also been rigorous mathematical analysis concerning the existence of solutions for the problem of cracks for such models by Bulicek et al. [17]. An interesting class of open problems that is worth studying is dynamic problems involving the propagation of cracks within this new class of constitutive equations. A class of problems that is connected with the study carried out here is the dynamics of a crack that is subject to sinusoidal oscillations similar to the problems studied in this paper in the absence of cracks.

Unlike the classical problem for the linearized elastic body wherein the problem under consideration would reduce to a single wave equation for the displacement field, in the current approach we have to solve a system comprising of the balance of mass, balance of linear momentum and the constitutive relation simultaneously for the density, stress and displacement fields. For the special problem under consideration, due to the use of semi-inverse solution sought, the problem reduces to solving a system of two equations simultaneously for the shear stress and the displacement field. While the equations can be manipulated to yield just one equation for the shear stress, such an equation is of higher order than the system of equations one has to solve within the context of the class of models considered here, and hence one has to worry about additional boundary and initial conditions. In fact, different additional conditions lead to different physical situations and hence it is more sensible to study the coupled system than to eliminate one of the equations (the details of this issue are discussed in the next section).