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On the influence of forcing on observed wave dispersion: A strategy for mitigating the effects of material dispersion

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ABSTRACT

In the context of wave propagation, wave dispersion is typically associated with properties of the propagation medium. Here we demonstrate the influence of forcing (over a spatially extended region) on the phase and group velocities of a wave field in both dispersive and nondispersive media. The influence of forcing on observed wave dispersion is relevant because it provides a means for overcoming the effects of material dispersion (such as the spreading of wave packets) in a broad range of wave physics, from electromagnetism to acoustics.

In spatial dimensions greater than one, wave dispersion[1] is, in general, not fully characterizable in terms of the dispersive properties of the propagation medium alone. The observed phase and group properties of mutually interfering plane progressive waves, and ultimately the nature of the energy transport, depend intimately on not only the frequency dependence of the material dispersion of the propagation medium, but also on the relative directions of the constituent wave components. One thing that becomes apparent when considering wave dispersion arising from the interaction of noncodirectional plane waves is that it is no longer sufficient to idealize the origin of the various wave components as coming from infinity independent of forcing, as is the case when one derives the dispersion relation from a homogenous (source free) wave equation. Rather, it becomes necessary to consider the forcing from which the waves arise. As an example, the dependence of wave dispersion on forcing is famously manifest in Lord Kelvin's classic solution for the wave field produced by a uniformly moving vessel on the surface of a body of deep water [1]. In this case, because the various plane wave components forced by the vessel do not travel codirectionally, the observed wave dispersion is not entirely explained by the well known dispersion relation for gravity waves derived from the source free set of linearized equations describing their behavior [2]. Rather, the observed wave dispersion depends on both the material dispersion (in this case the physics associated with the interplay between the inertia of the medium and the gravitationally mediated restoring force) and the nature of the forcing (in this case the speed of vessel).

Starting with a linearized wave equation we present an approach to dispersion analysis that incorporates the effects of forcing. It is shown that dispersive properties inherent to the forcing signal influence the observed wave dispersion in a predictable and quantifiable manner. It is proposed that these results can be exploited to compensate for material dispersion effects, and that they are particularly suited to applications involving wave energy/information transmission from phased array line sources into dispersive media.

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[1] In the context of wave phenomena, the term dispersion is often interpreted to refer specifically to the relation between frequency and wavenumber as derived through the homogenous (i.e. source free) wave equation. In the source free homogeneous case, the manner in which a wave disperses can be completely characterized through the material dispersive properties of the propagation medium. As the present discussion includes the effects of forcing on the nature of the observed wave propagation, the term “wave dispersion” implies a broader interpretation that encompasses the local observed phase and group velocities as well as the nature of energy propagation of a wave. This usage is more than semantic as in an experimental.

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To begin, we consider a generalized version of the inhomogeneous wave equation for the scalar field, ψ,

$$\left(\nabla^2 + \mathcal{L}\right)\psi = S, \tag{1}$$

where ∇^2 represents the spatial Laplacian operator, $S = S(\boldsymbol{r}, t)$ represents forcing[2], and $\boldsymbol{r}$ and t denote spatial and temporal coordinates, respectively. The quantity $\mathcal{L} = \mathcal{L}(\boldsymbol{r}, t)$ is a linear differential operator that characterizes the dynamic propagation of waves through the medium. While the medium is assumed to be isotropic and homogenous, it may, nevertheless, be dispersive. For example, the choice $\mathcal{L} = -\frac{1}{c^2}\partial_t^2$ is appropriate for describing a nonattenuating, nondispersive, homogenous, isotropic, free space medium of phase speed c. To include the effects of attenuation and dispersion, a choice for the operator $\mathcal{L}$ might be $\mathcal{L} = -\frac{1}{c^2}\ 1\ \partial_t^2 + \beta\partial_t\nabla^2$ (with β as a real constant, $\beta > 0$), and so on.

Applying the notation $\tilde{Y}$ to denote the time-space Fourier transform representation of the variable Y, and defining the quantity, $\mathcal{C}^2 \equiv \omega^2\ \tilde{\mathcal{L}}$, the Fourier transform of Eq. (1) yields an algebraic equation,

$$\tilde{\psi} = \frac{\tilde{S}}{(\omega/\mathcal{C})^2 - k^2}, \tag{2}$$

with k and ω denoting the wavenumber and angular frequency, respectively. The quantity $\mathcal{C}$ may be interpreted as representing a generalized complex valued medium phase speed that may be an algebraic function of frequency and wavenumber. Care must be taken in choosing the root of $\mathcal{C} \pm \omega\,\tilde{\mathcal{L}}^{1/2}$ so that it is consistent with causal nature of the physics of forced wave propagation. For the remainder of the discussion, only the positive root, corresponding to a positive medium phase velocity, is considered.

In this form, Eq. (2) is suggestive of the potential for forcing to influence the observed wave dispersion. For cases involving wavenumber independent forcing, $\tilde{S} = \tilde{S}(\omega)$, the observed wave dispersion can be fully described over all space and time through the quantity $\mathcal{C}$. In other words, all dispersive properties, including phase velocity, group velocity, and the nature of energy propagation, of wave field $\tilde{\psi}$ are dictated entirely by the properties of the propagation medium. The source free, homogenous case, $\tilde{S} = 0$, pertains to plane wave solutions. However, suppose more generally that the forcing is such that it imposes a relation between wavenumber and frequency, $\tilde{S} = \tilde{S}(\omega, k)$. In this case, it is no longer possible to fully describe the observed wave dispersion for all time and space through the quantity $\mathcal{C}$. In order to characterize the phase and group velocities, as well as the nature of energy propagation, the details of the forcing must also be accounted for.

One possibility of interest is the prospect of tailoring the forcing to mitigate certain undesirable effects of material dispersion. As will be demonstrated in the discussion that follows, by designing the forcing to include its own intrinsic dispersive properties, it is possible to predictably manipulate the phase, group, and energy transport characteristics of the generated wave field. The remainder of the discussion is dedicated to the presentation of examples that illustrate the influence of forcing on the observed wave field and how dispersive forcing[3] can be applied to mitigate the effects of material dispersion.

At this point, to simplify the mathematical treatment and facilitate the analysis, the discussion is restricted to the case of wave forcing from an infinite line source in a 2-dimensional isotropic, homogenous medium. While this restriction may seem contrived, it is actually quite relevant. Not only is it the geometry that describes the ocean waves generated by a vessel moving on the ocean's surface, it is a geometry that is useful for many seismic and ocean acoustic applications that involve line arrays along a surface boundary[4].

Consider a line source signal, $S(x,z,t)$, that is coincident with the z-axis in a 2-D Cartesian coordinate system, (x,z),

$$S(x,z,t) = \delta(x)\int dk_z A e^{ik_z(z - Vt)}, \tag{3}$$

where $\delta(x)$ denotes the Dirac-delta function with respect to coordinate x. Expressing the signal in this form, as a superposition of progressive sinusoids of amplitude A that propagate at phase velocity V, makes the dispersive properties of the signal straightforwardly explicit. More specifically $A = A(k_z)$ represents the (in general complex valued) coefficient of the corresponding constituent sinusoid of axial wavenumber k_z. By tailoring the axial phase speeds of the constituent sinusoids to vary as a function of the corresponding axial wavenumber, $V = V(k_z)$, it is possible to design a signal that exhibits desired dispersive properties. While it is perfectly legitimate to consider cases for which V is a complex number (leading to signal whose constituent sinusoidal components attenuate or amplify as they propagate), only the case of purely real V is considered here.

Substituting the line source signal of Eq. (3) into the 2-D version of the constitutive wave equation (Eq. (1)) and performing Fourier transforms over x, z, and t yields the 2-D expression of Eq. (2),

$$\tilde{\psi} = \frac{A\delta(\omega + k_z V)}{(\omega/\mathcal{C})^2 - k_x^2 - k_z^2}. \tag{4}$$

[2] For the purposes of the discussion, forcing can be thought of as an experimentally introduced and controllable aspect of the system: for example, an experimentally designed source signal transmitted via a phased array.

[3] In analogy to the terminology used to describe a propagation medium as either dispersive or nondispersive, it is useful for the purposes of the discussion to introduce the terms "dispersive forcing" and "nondispersive forcing" to distinguish a manner of forcing that is intrinsically dispersive from a manner of forcing that is not intrinsically dispersive.

[4] For the case of finite vessel trajectories or finite length arrays, the idealization of an infinite line source represents the high wavenumber (short wavelength) limit.