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# An exact expression for transient forced internal gravity waves in a Boussinesq fluid

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**Abstract**

An exact expression is derived for the propagation of transient forced internal gravity waves in a Boussinesq fluid with constant horizontal mean velocity. The horizontal wavelength of the forcing is assumed to be large relative to the vertical wavelength of the perturbation, and so the long-wave limit is taken. The solution consists of a part with steady amplitude and a transient part that goes to zero in the limit of infinite time. Two independent time scales are identified in the evolution of the transient term, one connected to the height of the wave above the source level and the other to the horizontal velocity of the background flow. The amplitude of the transient term is determined by the vertical component of the group velocity of the wave propagation. Because of the exact nature of the solution, it can be used as a starting point for further analytical and numerical studies of gravity-wave propagation.


*Keywords:* Internal gravity waves; Stratified flow; Boussinesq approximation

## 1. Introduction

Internal gravity waves are oscillations that occur in density-stratified fluids as a result of upward buoyancy forces and the restoring force of gravity [5]. Gravity-wave interactions have profound effects on the general circulation of the Earth's atmosphere and oceans and thus affect global weather and climate. Although such effects are in general highly nonlinear, some insight into the behaviour and propagation of gravity waves can be obtained through simple linear models. The equations that govern the evolution of gravity-wave propagation in a density-stratified fluid can be linearized if it is assumed that the wave amplitude is small relative to the magnitude of the large-scale flow. Under such assumptions, analytic solutions can sometimes be derived, but they are generally approximate. Less frequently, it is possible to obtain exact analytic solutions.

In a two-dimensional system, defined in terms of Cartesian coordinates in the horizontal and vertical directions, the background flow is often assumed to be a parallel (horizontal) flow with the background velocity being the horizontal mean of the total velocity. In regions of the atmosphere where the vertical variation of

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the background density is relatively small, one is justified in making the Boussinesq approximation, in which density variations with altitude are neglected in the acceleration terms, but retained only in the buoyancy force term in the vertical momentum equation [6].

An assumption frequently made in theoretical studies of atmospheric waves is that the waves are sinusoidal in time and in the horizontal direction and, thus, have a steady amplitude that depends only on altitude. With that assumption and the Boussinesq approximation, the resulting equations can be linearized to give a second-order ordinary differential equation, known as the Taylor–Goldstein equation, for the amplitude of the waves. Solutions of this equation are well known. For example, for a background flow with height-independent horizontal velocity, the solutions are exponential functions of altitude that may be either oscillatory or decay or grow with height.

However, observations of gravity waves in the atmosphere indicate that gravity-wave evolution is time-dependent and not generally strictly periodic [7]. A more realistic representation of gravity waves would be one that would allow the wave amplitude to change with time. In that case, the linearized Boussinesq equations lead to a fourth-order partial differential equation for the wave amplitude. In this paper, we derive a solution of this time-dependent equation for the case where the waves are forced at a given level in the lower atmosphere, for example, by topography or convection, and propagate upwards. The forcing is assumed to be sinusoidal in the horizontal direction, as well as in time, but the wave amplitude changes with time. By means of a Laplace transform, an exact analytic solution is obtained that describes the transient evolution of the waves.

Booker and Bretherton [3] derived an approximate time-dependent solution using Laplace transform techniques for a configuration with vertical shear, i.e., where the background horizontal velocity is a function of height. Since then, approximate analytic solutions have been derived for other configurations (see, for example, [2,5] or [8] for reviews). In this paper, we consider a configuration with a height-dependent background velocity profile and derive an exact time-dependent solution. The governing equations are simplified by taking the long-wave limit, in which the horizontal wavelength of the forcing is assumed to be large relative to the vertical wavelength of the perturbation. This is a reasonable assumption for gravity waves in the middle atmosphere, where horizontal wavelengths are typically tens to hundreds of kilometers, but vertical wavelengths can be as small as 2–5 km [4]. The solution consists of a part with steady amplitude, equivalent to the solution of the Taylor–Goldstein equation, and a transient part that goes to zero in the limit of infinite time. The details of the model are given in the next section and the solution procedure is described in Appendix A.

## 2. Analytic solution

The governing equations for this study are the equations of motion for a two-dimensional fluid defined in terms of Cartesian coordinates $x$ and $z$ in the horizontal and vertical directions, respectively, and a time variable $t$. With the Boussinesq approximation the continuity equation is simply

$$u_x + w_z = 0, \tag{1}$$

where $u$ and $w$ are the horizontal and vertical components of the velocity and the subscripts denote partial differentiation. This allows us to define a streamfunction by

$$u = -\Psi_z, \quad w = \Psi_x, \tag{2}$$

so that the horizontal and vertical momentum equations can be combined into a single equation for the streamfunction. Throughout this study we shall work with nondimensional variables and parameters. The spatial variables are made nondimensional on the basis of typical length scales $L$ and $H$ in the horizontal and vertical directions respectively, which are of the order of magnitude of the horizontal and vertical wavelengths of the waves under consideration.

The function $\Psi(x,z,t)$ in (2) is the total streamfunction; it is decomposed into a contribution from the steady basic flow and a time-dependent perturbation:

$$\Psi(x,z,t) = \bar{\psi}(z) + \varepsilon\psi(x,z,t). \tag{3}$$

The basic flow has velocity $(\bar{u}, 0)$, where $\bar{u}$ is constant, and its streamfunction is $\bar{\psi} = -\bar{u}z$. The total density is also decomposed into a steady basic part $\bar{\rho}(z)$ and a time-dependent perturbation $\varepsilon\rho(x,z,t)$. The parameter $\varepsilon$