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Computation of transmissions and reflections in geometrical optics via the reduced Liouville equation ☆

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Abstract

We develop a numerical scheme for the wave front computation of complete transmissions and reflections in geometrical optics. Such a problem can be formulated by a reduced Liouville equation with a discontinuous local wave speed or index of refraction, arising in the high frequency limit of linear waves through inhomogeneous media. The key idea is to incorporate Snell's Law of Refraction into the numerical flux for the reduced Liouville equation. This scheme allows a hyperbolic CFL condition, under which positivity, and stabilities in both l^∞ and l^1 norms, are established. Numerical experiments are carried out to demonstrate the validity and accuracy of this new scheme.

Keywords: Geometrical optics; Reduced Liouville equation; Level set method; Hamiltonian-preserving schemes; Snell's law of refraction

1. Introduction

In this paper, we construct and study a numerical scheme for the reduced Liouville equation in two space dimension:

$$f_t + c(x,y)\cos\theta f_x + c(x,y)\sin\theta f_y + [c_x\sin\theta - c_y\cos\theta]\, f_\theta = 0, \tag{1.1}$$

where $c(x,y)$ is the local wave speed, $f(t,x,y,\theta)$ is the density distribution of particles depending on position (x,y), time t and the slowness angle $\theta \in (-\pi,\pi]$. We are concerned with the case when $c(x,y)$ contains *discontinuities*, corresponding to different indices of refraction in different media. This discontinuity will generate an *interface*, crossing which waves will undergo transmissions or reflections.

The reduced Liouville equation (1.1) is obtained by using the constant Hamiltonian condition in the 2D full phase space Liouville equation

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$$f_t + \frac{c(x,y)\xi}{\sqrt{\xi^2+\eta^2}} f_x + \frac{c(x,y)\eta}{\sqrt{\xi^2+\eta^2}} f_y - c_x\sqrt{\xi^2+\eta^2} f_\xi - c_y\sqrt{\xi^2+\eta^2} f_\eta = 0, \tag{1.2}$$

where (ξ,η) is the slowness vector. The Liouville equation (1.2) is the phase space description of the Hamiltonian system with Hamiltonian

$$H(x,y,\xi,\eta) = c(x,y)\sqrt{\xi^2+\eta^2}. \tag{1.3}$$

In classical mechanics the Hamiltonian (1.3) of a particle remains a constant along the particle trajectory, even when it is being transmitted or reflected by the interface. By using the condition $H \equiv C$ for some constant C, one arrives at the reduced Liouville equation (1.1) which is computationally more efficient.

The Liouville equation (1.2) arises in the phase space description of geometrical optics. It is the high frequency limit of the linear wave equation

$$u_{tt} - c(\mathbf{x})^2\Delta u = 0, \quad t > 0, \ \mathbf{x} \in R^2. \tag{1.4}$$

Recently several phase space based level set methods are based on the equation (1.2) or (1.1), see [13,16,23,33,36]. It was used to compute the multivalued phase or velocity beyond caustics. The computations of multivalued solution in geometrical optics, or more generally for nonlinear PDEs, have been an active area of research in recent years, see [2,3,5,4,8,15,10,11,13,12,18,19,16,24,37,41,6,36,20,22,23]. However, all these works were developed without the interface. The analytical studies on the geometrical optics limit of linear wave equations through interfaces or solid boundaries were carried out in [1,32,39].

In our previous works, we constructed the *Hamiltonian-preserving schemes* that are suitable for the full phase space Liouville equation (1.2) with complete transmissions and reflections [26] or with partial transmissions and reflections [27]. The design principle there was to build the behavior of a particle at the interface – cross over with a changed velocity or/and be reflected with a negative velocity according to a constant Hamiltonian – into the numerical flux. See also earlier works [35,25]. It gives a selection criterion to select a unique solution, consistent to Snell's Law of Refraction, to the governing equation, which is linearly hyperbolic with singular (discontinuous or measure-valued) coefficients.

When only the wave front is interested one can just use the reduced Liouville equation which reduces the dimension by one. The key idea for the reduced Liouville equation is still to build into the numerical flux the wave behavior at the interface. This is given by Snell's Law of Refraction. This new, explicit scheme gives a method to select out the physically correct solution for the reduced Liouville equation (1.1) with singular coefficients, and like those in [25,26], it allows a typical hyperbolic stability condition $\Delta t = O(\Delta x, \Delta y, \Delta\theta)$, under which we also establish the positivity, and l^∞ and l^1 stability for the scheme.

The level set approach developed in [33,7,36], by using the reduced Liouville equation, has several advantages such as automatically handling the multivalued wave fronts and controlling the solution resolution. When the wave speed is smooth, the use of a standard finite difference method (SFDM) for the reduced Liouville equation, which is linearly hyperbolic, should be satisfactory. For problems with discontinuous wave speeds, one can still formally use the SFDM by either ignoring or smoothing out the wave speed discontinuities. However, as will be shown by numerical examples in Section 4, the use of the SFDM by ignoring the wave speed discontinuities typically does not lead to physically correct numerical solution that is consistent to Snell's Law. On the other hand, the use of the SFDM by smoothing out the wave speed discontinuities may give convergent solutions, but typically suffers from a severe CFL condition as well as poorer numerical resolution [33,26]. In [7] the authors take into account the wave reflection into the numerical scheme without considering wave transmission. See also a higher order method able to compute multiple reflections [9]. Such a consideration is suitable for waves hitting a solid wall. However, through an interface, waves can be reflected or transmitted, requiring the numerical scheme to be able to handle both situations. The scheme developed in this paper has such a capability.

We present and validate the scheme in the case of single interface. It can be applied to the case of multiple, isolated interfaces naturally. Moreover, Our idea is not restricted to two space dimension. It can be extended to three space dimension as well. However, as pointed out in [27], its generalization to the case of partial transmissions and reflections at the interface is not straightforward, and will be considered in our future study.