GENERALIZED SEPARATION OF VARIABLES FOR NONLINEAR EQUATION $u_{tt} = F(u)u_{xx} + aF'(u)u_x^2$

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We consider the equation $u_{tt} = F(u)u_{xx} + aF'(u)u_x^2$, where $F(u)$ is an arbitrary function and $a \neq 0$ is a constant. The problem in question is for which functions $F(u)$ this equation admits the ansatz $t = w_1(x)d(u) + w_2(x)$ reducing it to a system of two ordinary differential equations with unknown functions $w_1(x)$ and $w_2(x)$. New classes of exact solutions with generalized separation of variables were constructed for these equations, which cannot be obtained by the method of classical group analysis.

Keywords: nonlinear hyperbolic equations; generalized separation of variables.

1. Introduction

Recently much attention was paid to finding exact solutions of nonlinear equations of mathematical physics with separated variables. After generalization of the concept of variable separation, the range of studied equations was extended. In [1, 2] were described some types of parabolic and hyperbolic equations with quadratic nonlinearity that admit exact solutions with the generalized separation of variables of the form $u = \varphi(x)\psi(t) + \chi(t)$. R. Z. Zhdanov [3] described all nonlinear wave and heat equations of the form

$$u_{tt} + u_{xx} = f(u)$$

that admit exact solutions with the functional separation of variables,

$$u(x,t) = F(z), \qquad z = \varphi(x) + \psi(t).$$

New non-Lie solutions of a nonlinear wave equation were obtained by means of a generalized conditional-group approach to the functional separation of variables in [4].

There exists a connection between the generalized separation of variables and nonclassical symmetries (namely, Lie–Bäcklund symmetries). It was considered, in particular, in [5, 6] where essentially new separable solutions of some nonlinear evolution equations were obtained.

This paper is related to the class of equations discussed above. We consider the nonlinear equation

$$\frac{\partial^2 u}{\partial t^2} = F(u)\frac{\partial^2 u}{\partial x^2} + aF'(u)\left(\frac{\partial u}{\partial x}\right)^2, \qquad (1)$$

that appears in wave and gas dynamics and liquid crystals theory, and has a number of other applications. It is known that in the general case this equation has exact solutions,

$$u(x,t) = \omega(z), \qquad z = kx + \lambda t,$$

$$u(x,t) = \omega(\xi), \qquad \xi = \frac{x+b}{t+c},$$

where k, λ, b and c are arbitrary constants.

In the case $a = 1$ the group properties of Eq. (1) were studied in [7] by the method of S. Lie.

If $F(u) = \lambda u^k$, then Eq. (1) has an exact solution in the form of a product of functions of different arguments,

$$u = \varphi(x)\psi(t),$$

and if $F(u) = \lambda \exp(bu)$, then (1) has an exact solution as a sum of functions of different arguments,

$$u = \varphi(x) + \psi(t).$$

Qualitative analysis of the structure of solutions of Eq. (1) was performed in [8, 9].

An important case of Eq. (1), namely with $F(u) = \lambda u$, was considered in [10–12]. The method of constructing exact solutions of (1) of the form

$$u(x,t) = \sum_{i=1}^{k} f_i(t)a_i(x) \qquad (2)$$

was used in [10, 11]. This method is based on finding finite-dimensional subspaces that are invariant under the differential operator corresponding to the right-hand side of Eq. (1). Solutions of the form (2) for $k > 1$ are called the solutions of generalized separation of variables. For this case the ansatz

$$u = \sum_{i=1}^{m} w_i(t)a_i(x) + f(x,t), \qquad m \geq 1, \qquad (3)$$