GEOMETRY AND SHAPE OF MINKOWSKI'S SPACE CONFORMAL INFINITY

Arkadiusz Jadczyk

Center CAIROS, Institut de Mathématiques de Toulouse,
Université Paul Sabatier, 31062 TOULOUSE CEDEX 9, France
(e-mail: arkadiusz.jadczyk@cict.fr)

(*Received July 6, 2011*)

We review and further analyze Penrose's 'light cone at infinity'—the conformal closure of Minkowski space. Examples of a potential confusion in the existing literature about its geometry and shape are pointed out. It is argued that it is better to think about conformal infinity as of a needle horn supercyclide (or a limit horn torus) made of a family of circles, all intersecting at one and only one point, rather than that of a 'cone'. A parametrization using circular null geodesics is given. Compactified Minkowski space is represented in three ways: as a group manifold of the unitary group U(2), a projective quadric in six-dimensional real space of signature (4,2), and as the Grassmannian of maximal totally isotropic subspaces in complex four–dimensional twistor space. Explicit relations between these representations are given, using a concrete representation of antilinear action of the conformal Clifford algebra Cl(4,2) on twistors. Concepts of space-time geometry are explicitly linked to those of Lie sphere geometry. In particular conformal infinity is faithfully represented by planes in 3D real space plus the infinity point. Closed null geodesics trapped at infinity are represented by parallel plane fronts (plus infinity point). A version of the projective quadric in six-dimensional space where the quotient is taken by positive reals is shown to lead to a symmetric Dupin's type 'needle horn cyclide' shape of conformal infinity.

Keywords: Minkowski space, space-time, conformal group, twistors, infinity, Clifford algebra, cyclide, null geodesics, light cone, Lie sphere geometry.

Mathematics Subject Classification 2000: 83A05, 81R25, 53A30, 14M99.

1. Introduction

A persistent confusion about Minkowski's space conformal infinity started with a widely quoted paper by Roger Penrose *The light cone at infinity* [1]. In the abstract to this seminal paper Penrose wrote:

> From the point of view of the conformal structure of space-time, "points at infinity" can be treated on the same basis as finite points. Minkowski space can be completed to a highly symmetrical conformal manifold by the addition of a null cone at infinity—the "absolute cone".

He then elaborated in the main text:

> Let x^μ be the position vector of a general event in Minkowski space-time relative to a given origin O. Then the transformation to new Minkowskian coordinates $\hat{x}^\mu$ given by
>
> $$\hat{x}^\mu = \frac{x^\mu}{x_\alpha x^\alpha}, \qquad x^\mu = \frac{\hat{x}^\mu}{\hat{x}_\alpha \hat{x}^\alpha}, \tag{1}$$
>
> is conformal ("inversion with respect to O"). Observe that the whole null cone of O is transformed to infinity in the $\hat{x}^\mu$ system and that infinity in the x^μ system becomes the null cone of the origin $\hat{O}$ of the $\hat{x}^\mu$ system. ("Space–like" or "time-like" infinity become $\hat{O}$ itself but "null" infinity becomes spread out over the null cone of O.) Thus, from the conformal point of view "infinity" must be a null cone.

Penrose's statement, "*that infinity in the x^μ system becomes the null cone of the origin $\hat{O}$ of the $\hat{x}^\mu$ system*" apparently had a confusing effect even on some experts in the field. For instance, in the monograph [2, p. 127], we find the statement that "*'conformal infinity' is the result of the conformal inversion of the light cone at the origin of M*," and in another monograph Huggett and Tod write about the compactified Minkowski space M^c [3, p. 36]: "*Thus M^c consists of M with an extra null cone added at infinity.*" Not only they write so in words, but they also miss a part of the conformal infinity (the closing two–sphere) in their, otherwise excellent and clear, formal analysis.

This apparent confusion has been described in [4], where also a deeper analysis of the structure of the conformal infinity has been given using, in particular, Clifford algebra techniques. In [5] a close similarity has been noticed between the geometry and shape of the conformal infinity with that of Dupin's type (super)cyclide. In the present paper we review and develop these ideas further on, and also make a step in relating them to Lie sphere geometry in $\mathbb{R}^3$ developed by Sophus Lie [7], Wilhelm Blaschke [8] and Thomas E. Cecil [9].

In Section 1 we introduce the compactified Minkowski space M^c (via Cayley's transform) following Armin Uhlmann [10], as the group manifold of the unitary group $U(2)$, and the conformal infinity as the subset of $U(2)$ consisting of those matrices $U \in U(2)$ for which $\det(U - I) = 0$. In Section 3 we review the relation of the compactified Minkowski space and its conformal infinity part to the group $SU(2,2)$ (the spin group of the conformal group), and to its action on $U(2)$ via fractional linear transformations $U' = (AU + B)(CU + D)^{-1}$. In particular, the role of totally isotropic subspaces of $\mathbb{C}^{2,2}$ (as null geodesics and as points of M^c) is elucidated there. In Section 4 the $SU(2,2)$ formalism is explicitly related to the $O(4,2)$ representation via a particular matrix realization (by antilinear transformations) of the Clifford algebra $Cl_{4,2}$. The main results of this section are contained in Proposition 1 and Corollary 1, where an explicit formula for a bijective map between the projective quadric of $\mathbb{R}^{4,2}$ and $U(2)$ is given—cf. Eq. (5). Our conventions are: coordinates x^μ, $\mu = 1, \ldots, 4$, with $x^4 = ct$, for the Minkowski space, x^α, $\alpha = 1, \ldots, 6$ for $\mathbb{R}^{4,2}$ endowed with the quadratic form $Q(x) = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2 + (x^5)^2 - (x^6)^2$.