A NOTE ON EFFECT ALGEBRAS AND DIMENSION THEORY OF AF C*-ALGEBRAS

ANNA JENČOVÁ and SYLVIA PULMANNOVÁ

Mathematical Institute, Slovak Academy of Sciences,
SK-814 73 Bratislava, Slovakia
(e-mails: pulmann@mat.savba.sk, JENCA@mat.savba.sk)

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We continue in the investigation of the relations between effect algebras and AF C*-algebras started by Pulmannová, 1999. In particular, in analogy with the notion of a dimension group, we introduce the notion of a dimension effect algebra as an effect algebra obtained as the direct limit of finite effect algebras with RDP. We also give an intrinsic characterization of dimension effect algebras. It turns out that every dimension effect algebra is a unit interval in a dimension group with a unit. We prove that there is a categorical equivalence between the category of countable dimension effect algebras and unital AF C*-algebras.

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1. Introduction

In the traditional approach, the mathematical description of a quantum mechanical system is based on a Hilbert space. Experimentally verifiable propositions about a physical system are modeled by projections on the corresponding Hilbert space.

Recently, to describe unsharp propositions and measurements, new abstract partial algebraic structures were introduced, called effect algebras (alternatively, D-posets). In the Hilbert space approach, the effect algebra is realized by self-adjoint operators lying between zero and identity, which are called the Hilbert space effects. The transition from projections to effects corresponds to the transition from projection-valued (PV) to positive operator-valued (POV) observables. The latter play an important role in the theory of quantum measurements [6].

For the systems with infinitely many degrees of freedom, the C*-algebraic approach is more appropriate [18]. For example, the physical system arising as the thermodynamic limit of finite spin-like systems is described by AF C*-algebras, i.e. the norm closure of an ascending sequence of finite-dimensional C*-algebras.

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Among general effect algebras, a special role is played by MV-algebras, introduced as an algebraic description of many-valued logic [7, 8]. In [22], a correspondence was shown between (countable) MV-algebras and the subclass of unital AF C*-algebras, whose Murray–von Neumann order is a lattice. This result was extended in [24] to all unital AF C*-algebras, and a certain subclass of effect algebras with the Riesz decomposition property, extending the class of MV-algebras.

In the present paper, we give an intrinsic characterization of the latter subclass of effect algebras. We also show that they can be obtained as direct limits of sequences of finite effect algebras with the Riesz decomposition property. In analogy with the notion of dimension groups, we introduce the notion of dimension effect algebras. It turns out that dimension effect algebras are exactly the unit intervals in unital dimension groups. We prove a categorical equivalence between countable dimension effect algebras and unital AF C*-algebras. A similar result was stated in [24, Theorem 8] without proof.

2. Preliminaries on K_0-theory of AF C*-algebras

The purpose of dimension theory of C*-algebras is to measure the "dimensions" of projections in the algebra. The dimension of a projection in a matrix algebra is a nonnegative integer, while in a finite von Neumann factor one obtains a nonnegative real number. In a C*-algebra A, in general, the dimension function must be given by values in a pre-ordered abelian group (a so-called K_0 group), rather than in real numbers. For basics of K_0-theory for C*-algebras see e.g. [4, 17, 27, 28].

Given a C*-algebra A, two projections $p, q \in A$ are equivalent, written $p \sim q$, if there is a partial isometry u such that $u^*u = p$ and $uu^* = q$. Let $\mathrm{Proj}(A)$ denote the set of all equivalence classes of projections of A. A partial binary operation $+$ can be defined on $\mathrm{Proj}(A)$ in the following way: for two equivalence classes $[p]$ and $[q]$ their "sum" $[p] + [q]$ exists iff there are representatives $p' \in [p]$ and $q' \in [q]$ with $p'q' = 0$, in which case $[p] + [q] = [p' + q']$.

Recall that in the K_0-theory of C*-algebras, the definition of $K_0(A)$ for a C*-algebra A requires simultaneous consideration of all matrix algebras over A. Denote by $\mathcal{M}_n(A)$ the set of all $n \times n$ matrices with entries in A. Let $\mathcal{M}_\infty$ denote the algebraic direct limit of $\mathcal{M}_n(A)$ under the embedding $a \mapsto \mathrm{diag}(a, 0)$. Then $\mathcal{M}_\infty(A)$ can be thought as the algebra of all infinite matrices with only finitely many nonzero entries. Let $V(A) := \mathrm{Proj}(\mathcal{M}_\infty(A))$. If A is separable, then $V(A)$ is countable. There is a binary operation on $V(A)$: If $[e], [f] \in V(A)$, choose $e' \in [e], f' \in [f]$ with $e' \perp f'$. Notice that this is always possible by "moving down" the diagonal. Then define $[e] + [f] = [e' + f']$. This operation is well defined and makes $V(A)$ into an abelian semigroup with the additive zero $[0]$.

If A is a unital C*-algebra, then $K_0(A)$ is defined as the Grothendieck group for $V(A)$. For nonunital C*-algebras the construction of K_0 is more refined. The embedding of $V(A)$ into $K_0(A)$ is injective iff $V(A)$ has cancelation, i.e. if $[e] + [g] = [e] + [h]$ implies $[g] = [h]$ for $[e], [g], [h] \in V(A)$. $K_0(A)$ can be pre-ordered taking the image of $V(A)$ in $K_0(A)$ as $K_0(A)^+$. We can also define the