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CRAMER RULE FOR QUATERNIONIC LINEAR EQUATIONS IN QUATERNIONIC QUANTUM THEORY*

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By means of a complex representation of a quaternion matrix and a companion vector, this paper introduces a new definition of determinant for a quaternion matrix, derives a technique of finding an inverse matrix of a quaternion invertible matrix, and gives a Cramer rule for quaternionic linear equations in quaternionic quantum theory.

Keywords: Cramer rule, complex representation, companion vector, quaternionic linear equation.

1. Introduction

In the study of quaternionic quantum mechanics and some other applications of quaternions [1–6], one often encounters the problem of solutions of quaternionic linear equations. Because of noncommutativity of quaternions, the solutions of quaternionic linear equations are more difficult. In papers [7, 8], by means of a complex representation and a companion vector, we have studied the problems of solutions of quaternionic linear equations and algorithms for eigenvalues and eigenvectors of a quaternion matrix, respectively, in quaternionic quantum theory. The paper [7] gave not only the necessary and sufficient conditions for the existence of solutions of quaternionic linear equations, but also a technique of finding solutions to quaternionic linear equations. This paper, by means of a complex representation and a companion vector, introduces a new definition of determinant for a quaternion matrix, gives an algorithm for inverse matrix of a quaternion invertible matrix, and gives Cramer rule for quaternionic linear equations in the quaternionic quantum theory.

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{C} = \{a + b\sqrt{-1} | a, b \in \mathbb{R}\}$ the field of complex numbers, and $\mathbb{Q}$ the field of quaternions. For $x \in \mathbb{C}$, $\overline{x}$ is the conjugate of x. Let $\mathbb{F}^{m\times n}$ denote the set of $m \times n$ matrices on a field $\mathbb{F}$. For any $A \in \mathbb{C}^{n\times n}$, A^T, $\det(A)$ and $\operatorname{adj}(A)$ denote the transpose, the determinant and the adjoint of the matrix A, respectively.

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We first recall the complex representation and the companion vector [7, 8]. For any quaternion $x = x_0 + x_1 i + x_2 j + x_3 k = y + zj \in \mathbb{Q}$, in which $x_i \in \mathbb{R}$, and $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, and a quaternion matrix $A \in \mathbb{Q}^{m\times n}$, the complex representations of x and A were defined respectively by

$$x^f = \begin{bmatrix} x_0 + x_1\sqrt{-1} & x_2 + x_3\sqrt{-1} \\ -x_2 + x_3\sqrt{-1} & x_0 - x_1\sqrt{-1} \end{bmatrix} = \begin{bmatrix} y & z \\ -\overline{z} & \overline{y} \end{bmatrix} \in \mathbb{C}^{2\times 2}, \tag{1.1}$$

and

$$A^f = (a_{st}^f) = \left[\begin{bmatrix} y_{st} & z_{st} \\ -\overline{z}_{st} & \overline{y}_{st} \end{bmatrix}\right] \in \mathbb{Q}^f \subseteq \mathbb{C}^{2m\times 2n}. \tag{1.2}$$

If $\alpha = (x_1, x_2, \ldots, x_{2n})^T \in \mathbb{C}^{2n\times 1}$, then the companion vector α^c of a vector α was defined as $\alpha^c = (-\overline{x}_2, \overline{x}_1, -\overline{x}_4, \overline{x}_3, \ldots, -\overline{x}_{2n}, \overline{x}_{2n-1})^T \in \mathbb{C}^{2n\times 1}$.

For any quaternion matrix $A \in \mathbb{Q}^{m\times n}$, by the definition of complex representation of a quaternion matrix there exist complex vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $A^f = (\alpha_1, \alpha_1^c, \alpha_2, \alpha_2^c, \ldots, \alpha_n, \alpha_n^c)$.

From [7, 8] we know that if $A \in \mathbb{Q}^{m\times n}$ and $B \in \mathbb{Q}^{n\times s}$, then

$$(AB)^f = A^f B^f, \tag{1.3}$$

and if $A \in \mathbb{Q}^{n\times n}$, then A is nonsingular if and only if A^f is nonsingular, and $(A^f)^{-1} = (A^{-1})^f$. Let $A \in \mathbb{Q}^{n\times n}$, then the real eigenvalues of the complex representation A^f appear in pairs, and the imaginary eigenvalues of A^f appear in conjugate pairs, respectively.

The following result follows immediately from the statement above.

PROPOSITION 1.1. *Let $A \in \mathbb{Q}^{n\times n}$ and A^f be its complex representation. Then the determinant* $\det(A^f)$ *of A^f is a nonnegative real number.*

2. Determinant of quaternion matrices

In this section we define a determinant of a quaternion matrix by means of its complex representation and companion vector.

DEFINITION 2.1. Let $A \in \mathbb{Q}^{n\times n}$ and A^f be its complex representation. Then *determinant* $\det(A)$ *of the quaternion matrix A* is defined as

$$\det(A) = \det(A^f). \tag{2.1}$$

From this definition and Proposition 1.1 we easily know that the determinant of a quaternion matrix is a nonnegative real number, and if $A \in \mathbb{Q}^{n\times n}$, $B \in \mathbb{Q}^{n\times n}$, then

$$\det(AB) = \det(A)\det(B). \tag{2.2}$$

Let $A \in \mathbb{Q}^{n\times n}$. Then it is easy to show by a direct calculation that $\operatorname{adj}(A^f) \in \mathbb{Q}^f$. Now we introduce the concept of adjoint matrix of a quaternion matrix A by means of complex representation.