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Analytical solutions of first-mode sloshing in new axisymmetric containers

Patrick Weidman

Department of Mechanical Engineering, University of Colorado, Boulder, CO 80309-0427, United States

HIGHLIGHTS

- Three new families of containers for axisymmetric mode-one sloshing.
- Analytic solutions for depth-dependent frequencies.
- Comparison with planar analogue geometries.
- Limiting cases of isochronous containers determined.

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ABSTRACT

A new family of axisymmetric containers denoted as D and two new series of families of axisymmetric containers denoted A_n and B_n are found for first-mode sloshing. Each member depends on streamfunction separation constant β. The containers are of annular shape and the frequency distributions are obtained in analytical form. The D family exists in the range $0.5 \le \beta < \infty$, involves logarithmic behavior, and exhibits a limiting shape in the form of an annular 90° wedge at low depth. The A_n container series which exist in the range $0 \le \beta \le \infty$ vary from cylindrical annulii at $\beta = 0$ to axisymmetric isochronous shapes for $\beta \to \infty$. The B_n container series exist in the range $\beta_n \le \beta \le \infty$ are more interesting in that the shapes for the minimum values β_n for existence of solutions in each family take the form of an annular 90° wedge at low depth and evolve to constant width at high depth. The shapes for all other values of $\beta > \beta_n$ are of hyperbolic form at low depth and also evolve to constant width at high depth. The limiting geometries at each n found as $\beta \to \infty$ correspond to axisymmetric isochronous containers.

1. Introduction

Sloshing in containers of various geometries has practical application and is of mathematical interest. Early treatises on the subject may be found in Lamb [1], Moiseev [2], Moiseev and Petrov [3] and recent monographs are given by Ibrahim [4] and Faltisen and Timokha [5].

In this paper we search for new axisymmetric containers that support first-mode sloshing about nodal circles. To place the present investigation in proper perspective we review relevant papers on planar and axisymmetric sloshing.

Recently, Weidman [6] reported three new families of containers for first-mode planar sloshing. These families, denoted $C1$, $C2$ and $C3$, are parameterized by streamfunction separation constant B. Containers $C1$ exist for $1 \le B \le \infty$, containers $C2$ exist for $0 \le B \le 1$ and containers $C3$ exist for $0 \le B \le \infty$. There is an affinity between the $C1$ and $C3$ families in that the limiting container profiles, found as $B \to \infty$, take the shape of the mode-one isochronous container. While the $C1$

E-mail address: weidman@colorado.edu.

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and $C3$ containers rise to infinite height, the $C2$ family of containers are of finite height. Also there is an affinity between the $C1$ and $C2$ families in that their $B = 1$ solutions both asymptote to 90° wedges at low height. The work presented here represents the axisymmetric analogue of this work.

Using the inverse method of solution Sen [7], *inter alia*, found standing waves in a number of three-dimensional containers including a hyperbolic paraboloid and a rectangular parallelepiped basin of arbitrary length and width. Also employing the inverse method, Troesch [8] searched for axisymmetric containers by assuming the velocity potential

$$\phi(r, \theta, z) = F(r, z)\cos m\theta \tag{1.1}$$

and choosing polynomial forms for $F(r, z)$ to compute streamlines, thus obtaining possible container shapes. Many interesting container shapes were found, including conical and hyperboloid containers with oscillations about m nodal diameters or a nodal circle. In these cases there was some algebraic variation of frequency with depth. More elaborate choices of the polynomials require computation of the container shape by integration of ordinary differential equations in parametric form. In this manner a very interesting axisymmetric geometry with nonmonotonic variation of height with radius was found for oscillations about a nodal circle.

In a follow on study, Troesch and Weidman [9] reported on planar and axisymmetric isochronous container shapes. Isochronous containers are those for which a frequency of a given mode of oscillation is invariant with liquid depth. Relevant to this study is the first-mode planar isochronous container whose shape is given as

$$z(x) = -\frac{1}{k}\log(\cos kx) \tag{1.2}$$

where $k = \omega^2/g$. Axisymmetric containers for oscillations about $m \geq 1$ nodal diameters were also found. These shapes are given in integral form as

$$z_m(r) = \frac{1}{\lambda}\int_0^{\lambda r} \frac{J_m(\rho)}{J'_m(\rho)}d\rho. \tag{1.3}$$

Weidman [10] extended the work of Troesch and Weidman [9] to include the case $m = 0$ for isochronous sloshing in axisymmetric containers about nodal circles. An infinite series of isochronous containers of annular form was found.

Some frequency results relevant to this study are for first-mode sloshing in a rectangular tank, a 90° wedge, a hyperbolic basin, and the isochronous container. For a rectangular tank the first-mode frequencies as a function of liquid depth H given by Lamb [1, §257] are

$$\frac{\omega^2}{g} = k\tanh(kH) \tag{1.4}$$

where k is a wavenumber which determines the asymptotic width of the tank. Frequencies for the first sloshing mode in a 90° wedge found by Lamb [1, §258] are given as

$$\frac{\omega^2}{g} = \frac{1}{H} \tag{1.5}$$

for which the free surface is flat. The isochronous container (1.2) has frequency

$$\frac{\omega^2}{g} = k. \tag{1.6}$$

The final planar container, only recently reported by Roberts [11], takes the form of a hyperbola with first-mode sloshing frequencies given by

$$\frac{\omega^2}{g} = \frac{1}{z_0 + H} \tag{1.7}$$

where z_0 is chosen to place the bottom of the hyperbola at $z = 0$. It is interesting to note that this simple solution is not found in Lamb [1] nor in the monographs by Ibrahim [4] and Faltisen and Timokha [5].

As mentioned above, the present investigation extends the work of Weidman [6] to find new families of first-mode sloshing in axisymmetric containers for which the container shapes and frequencies of the fundamental (most dangerous) oscillation mode are found in analytical form for potential functions separable in the (r, z) coordinates.

The presentation is as follows. An outline of the theoretical methodology is given in Section 2. The first family of container shapes is derived in Section 3. This is followed by two new series of families of containers presented in Sections 4 and 5. A discussion and concluding remarks are given in Section 6.

2. Mathematical methodology

Cylindrical coordinates (r, z) with coordinate velocities (u, w) are employed. The potential function $\mathbf{u} = \nabla\Phi$ for linear oscillations at frequency ω of an inviscid, incompressible, irrotational fluid in container V with free surface S is of the form

$$\Phi(r, z, t) = \phi(r, z)\cos\omega t. \tag{2.1a}$$