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Multicanonical analysis of rogue wave probabilities

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HIGHLIGHTS

- The multicanonical algorithm is found to predict accurately the probability of extremely rare rogue wave events.
- A simple parametrization for the rogue wave amplitude as a function of nonlinearity is presented.
- A rapid Runge–Kutta algorithm is developed for rogue wave propagation.

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ABSTRACT

The multicanonical procedure is applied to the nonlinear Schrödinger equation in conjunction with a high order finite difference solution procedure to determine the probability distribution function (pdf) of rogue wave power and heights. The analysis demonstrates a logarithmic dependence of the slope of the pdf on the nonlinearity coefficient at large heights.

1. Introduction

This paper presents a novel procedure for evaluating the probability distribution function (pdf) of rogue wave features. Rogue waves, which have been extensively studied in e.g. optical systems and deep ocean environments, are waves of extreme height that can be generated or influenced by numerous physical mechanisms, such as odd-order dispersion and nonlinearity [1–3] However, the unperturbed or perturbed nonlinear Schrödinger equation adequately describes the interaction between second order dispersion and nonlinearity that leads to rogue wave behavior in many situations and will therefore be employed in this paper. The pdf of e.g. the rogue wave heights then still depends on numerous parameters of the initial field such as its average power and spectral half width and is therefore difficult to quantify. This paper accordingly first introduces a rapid algorithm for evolving the acoustic or optical field based on a 4th order Runge–Kutta method together with a high-order centered finite difference approximation for the dispersion term (RKHD) that is shown to be generally more efficient than e.g. the Fourier transform based Runge–Kutta interaction procedure (RKIP) of [4]. Distribution functions of wave heights for various input powers and bandwidths are evaluated by employing both the standard Monte Carlo (MC) and multicanonical (MCMC) sampling [5]. The results of these simulations suggest a simple logarithmic parametrization relating the slope of the pdf to the nonlinearity coefficient in the rogue wave region as well as a greater generation probability

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Table 1
Parameter scaling.

	Scaled z, z'	Unit of z	Scaled t, t'	Unit of t	Scaled A, A'	Unit of A
Optical	$\frac{1}{1.04}\ \text{km}^{-1}\cdot z$	km	$\frac{1}{2}ps^{-1}\cdot t$	ps	$\sqrt{\frac{52}{5\gamma}}\frac{1}{10}\sqrt{W^{-1}}\cdot A$	$\sqrt{W}$
Oceanic	$2k_0\left(\frac{\upsilon\cdot\Delta\omega}{\omega_0}\right)^2\cdot z$	m	$\upsilon\cdot\Delta\omega\cdot t$	s	$\frac{k_0\omega_0}{\upsilon\Delta\omega\sqrt{2\gamma}}\cdot A$	m

for rogue waves in the presence of initial fields dominated by lower frequency components. Additionally, we confirm earlier observations that under some conditions, the oceanic rogue wave probability can increase as the nonlinearity is lowered.

2. Physical model

Rogue wave propagation is commonly modeled with the nonlinear Schrödinger equation in which the time and space variables are scaled differently in optical and ocean contexts. In optics, in terms of the complex field $A(z, t)$, which is periodic over the computational window length (CW), this equation, which preserves the power $|A|^2$ integrated over the CW, is given by [1]

$$\frac{\partial A}{\partial z} = i\frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \beta_3\frac{\partial^3 A}{\partial t^3} + i\gamma\,|A|^2 A \tag{1}$$

where the distance, time and computational window variables are given in Table 1. The nonlinear Schrödinger equation is then obtained by setting the third order dispersion coefficient, β_3, to 0.

In ocean contexts for which third order dispersion is typically neglected, the scaling is instead performed as in Table 1 according to [2], where ω_0 represents the carrier frequency, $k_0 = \omega_0^2/g$ is the wavenumber, $\Delta\omega$ is the half width of the frequency spectrum of the field and the scale factor is given in terms of the oceanic signal duration OCD by $\upsilon = CW/(OCD\cdot\Delta\omega)$.

In the optical calculations of this paper, the propagation length and signal duration (OPD) are set to 7.69 km and 102.4 ps. Different values of the initial power are modeled by appropriately scaling γ. In the oceanic case, the propagation length and oceanic signal duration (OCD) equal 3.46 km and 480 s. In both cases, the propagation length and signal duration correspond to a scaled propagation length $z' = 8$ while the computational window widths t' (CW) are set to 51.2, the integral of $\left|A'\right|^2$ over the computational window equals 256 and the third order dispersion term is omitted.

3. Computational method

3.1. Initialization and measurement

In the optical context, we first elucidate several general properties of the distribution function by considering an initial field described by the sum of ten monochromatic waves according to

$$A(z=0,t) = \sqrt{\frac{5\gamma}{52}}10\sqrt{W}\sum_{n=1}^{10}\cos(\omega_n t + \varphi_n) \tag{2}$$

$$A(z=0,t) = \sqrt{\frac{5\gamma}{52}}10\sqrt{W}\sum_{n=11}^{20}\cos(\omega_n t + \varphi_n) \tag{3}$$

with frequencies, $\omega_n = \frac{2\pi}{CW}\cdot n$. We then employ the multicanonical MCMC procedure in [5] to determine the pdf of the scaled wave power at a fixed point in the CW, $|A(z=8, t=0)|^2$ or the pdf of the maximum of the scaled wave power in the CW, $\max_{t\in CW}|A(z=8,t)|^2$, after a propagation distance of $z = 8$ scaled distance units. The Markov chain in these calculations is composed of the 10 (9 in the calculations of the pdf of the scaled maximum wave power that employ periodic boundary conditions) phase variables of (2). The magnitude of the variations in the phase variables between successive iterations is chosen such that the results are both in optimal agreement with those of the standard MC method in the high probability region and are nearly invariant when this magnitude is changed slightly in the low probability regions. To attain a statistically invariant steady-state distribution, the nonlinear Schrödinger equation was employed to propagate the field beyond five soliton periods (2.5π) [6] and the wave power was then sampled at $z = 8$, following a procedure often applied in oceanic propagation.

In oceanic propagation, we instead employ the Joint North Sea Wave Project spectrum (JNSWP) ((2) in [7, p. 160]) to construct a field with the energy spectrum

$$E(f) = \alpha g^2(2\pi)^{-4}f^{-5}\exp\left[-\frac{5}{4}\left(\frac{f}{f_{peak}}\right)^{-4}\right]\chi^{\exp\left[-\frac{1}{2}\left(\frac{f-f_{peak}}{\sigma f_{peak}}\right)^2\right]} \tag{4}$$