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Rayleigh limits for effective wavenumbers of randomly distributed porous cylinders. Comparison of explicit and implicit methods

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HIGHLIGHTS

- Low concentrations of porous cylinders at low frequencies are considered.
- Wavenumbers from explicit and implicit formulas are compared.
- Effective mass densities and bulk moduli are given.

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ABSTRACT

The effective wavenumber of the coherent wave propagating in a fluid containing parallel porous cylinders randomly distributed in space is derived at the Rayleigh limit for (i) explicit formulas: Independent Scattering Approximation (ISA), Waterman and Truell (WT) and Linton and Martin (LM) and (ii) for implicit formulas: Coherent Potential Approximation (CPA) and Generalized Self Consistent Method (GSCM) applied to WT and to LM. The effective mass density and bulk modulus are also derived. The validity of all the effective quantities is checked by recovering, when the porosity of the scatterers tends to zero, the case of an inhomogeneous medium of elastic cylinders.

1. Introduction

The problem of sound propagation through homogeneous compressible fluid or elastic domains containing randomly distributed scatterers has received a large attention in recent decades [1–5]. In the case of fluid domains, the scatterers can be as well bubbles, hard grains, elastic rods or contrast agents. The main interest lies on the description of the coherent wave which represents a statistical average over all possible configurations of the scatterers. The effective wavenumber k_{eff} of the coherent wave which is also called the effective wavenumber is complex-valued.

Several explicit formulas (i.e., which do not need to be solved) have been proposed for the calculation of k_{eff} in the case of cylindrical scatterers, among which the Independent Scattering Approximation (ISA) formula [6] derived by using Foldy's closure assumption [4]

$$k_{eff}^2 = k_{ISA}^2 = k_0^2 - 4in_0 f\,(0)\,, \qquad (1)$$

the Waterman and Truell [7] formula,

$$k_{eff}^2 = k_{WT}^2 = \left[k_0 - \frac{2in_0}{k_0} f\,(0)\right]^2 - \left[\frac{2in_0}{k_0} f\,(\pi)\right]^2, \qquad (2)$$

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Table 1
Physical constants of the porous medium and of saturating water. The function $F(\chi)$ is dynamic viscosity factor [16] and ω angular frequency.

Bulk modulus of grains	K_s
Dried frame bulk modulus	K_b
Dried frame shear modulus	μ
Density of grain	ρ_s
Water sound velocity	c_0
Density of water	ρ_0
Saturating water kinematic viscosity	η
Porosity	β
Porous frame permeability [15]	k (depends on β)
Mean pore radius [15]	a_p (depends on β)
Tortuosity (structure factor) [16]	α (depends on β)
Dynamic tortuosity	$\tilde{\alpha} = \alpha + i\beta\eta F(\chi)/(\omega k\rho_0)$, $\chi = \left(\omega a_p^2/\eta\right)^{1/2}$ ($F(\chi) \approx 1$ at low frequency)
Bulk modulii of frame	$H = \frac{(K_s-K_b)^2}{K_s(1+\beta(K_s/K_f-1))-K_b} + K_b + \frac{4}{3}\mu$ $C = \frac{K_s(K_s-K_b)}{K_s(1+\beta(K_s/K_f-1))-K_b}$

and the Linton and Martin [8,9] formula which is based on the closure assumption often called the quasicrystalline approximation (QCA) of Lax,

$$k_{eff}^2 = k_{LM}^2 = k_0^2 - 4in_0 f(0) + \frac{8n_0^2}{\pi k_0^2}\int_0^\pi \cot\left(\frac{\theta}{2}\right)\frac{d}{d\theta}[f(\theta)]^2\, d\theta. \tag{3}$$

In the foregoing, $f(\theta)$ is the far-field scattered amplitude of each cylinder in the direction θ, n_0 the number of scatterers per unit area and k_0 the wavenumber in the homogeneous compressible fluid host. The accuracy of the three formulas depends on the value of the concentration $c = n_0\pi a^2$ where a is the radius of circular cylinders, i.e., on the ratio n_0/k_0^2. Eqs. (1)–(3) were obtained under the assumption $n_0/k_0^2 \ll 1$. ISA contains terms of order zero and one in n_0/k_0^2; both WT and LM formulas contain the same orders plus a term of order two in n_0/k_0^2 and are therefore more accurate. But as shown by Derode et al. [10] both ISA and WT fail at high concentrations approaching 15% while LM still provides good results.

Implicit methods have been also used which include the Coherent Potential Approximation (CPA) [1,2,11] and the General Self Consistent Method (GSCM) of Yang and Mal [12] which originates from a self consistent scheme applied to the Waterman and Truell's formula. The basic idea of implicit methods consists in the replacement of the random medium by an effective medium with a complex wavenumber k_{eff} calculated self-consistently by imposing that the scattering arising from the local substitution of the effective medium by the actual medium (cylinder coated with a fluid shell) should vanish. In such methods, the wavenumber k_{eff} is obtained by solving an equation containing the far-field scattered amplitude.

In the present paper, the scatterers are made of a fluid saturated porous material that obeys Biot's theory [13,14]. A fast, a slow and a shear waves are assumed to propagate being all dispersive and attenuated. The scatterers are embedded in the same fluid as the saturating one. In Section 2, for long wavelengths, formulas for the effective wavenumbers of randomly distributed porous cylinders are derived up to the second order in concentration from which velocity $\omega/\mathrm{Re}\left(k_{eff}\right)$ and attenuation $\mathrm{Im}\left(k_{eff}\right)/\mathrm{Re}\left(k_{eff}\right)$ can be studied. At first, the Rayleigh limits for the explicit effective wavenumbers k_{ISA}, k_{WT} and k_{LM}, Eqs. (1)–(3), are obtained. One checks the results by comparing them with the limiting case of elastic scatterers (absence of porosity). In Section 3, one makes the equivalent derivations for the implicit effective wavenumbers, namely k_{CPA}, k_{G-WT} and k_{G-LM}. It is shown that the implicit methods differ from the explicit ones from the second order in concentration for a random medium of porous scatterers as well as for a random medium of elastic ones. In Section 4, the expressions found for the explicit and implicit effective wavenumbers are used to derive effective dynamic mass densities and bulk moduli. A comparative study between the different methods is provided.

2. Wavenumbers from explicit theories

Let us consider circular porous cylinders of radius a immersed in and saturated by a fluid of mass density ρ_0 and of sound velocity c_0. Whereas the fluid is assumed perfect out of the cylinders, it is viscous inside the pores, with a kinematic viscosity η. The incident longitudinal wave from the fluid has a wavenumber denoted by $k_0 = \omega/c_0$ where ω is the angular frequency; when hitting the scatterers, some penetrates and is converted into three poroelastic waves [13,14] with respective complex wavenumbers $\ell_1 = \omega/c_1$ (fast longitudinal wave), $\ell_2 = \omega/c_2$ (slow longitudinal wave) and $\ell_t = \omega/c_t$ (shear or transverse wave), before being reemitted into the fluid. The quantities c_1, c_2 and c_t denote complex velocities. The three wavenumbers ℓ_1, ℓ_2 and ℓ_t depend on a great number of parameters summarized in Table 1.

This paper deals with the low frequency range. The magnitudes of k_0 and $\left|\ell_j\right|$ $(j = 1,\ 2,\ t)$ being of the same order, the low frequency assumption is to consider normalized frequencies such that $k_0 a \ll 1$ and $\left|\ell_j a\right| \ll 1$. The far-field scattering