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The instability of Wilton ripples

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HIGHLIGHTS

- Numerical solutions to Euler's equations for periodic gravity–capillary waves.
- A variant of the boundary integral method for traveling wave solutions is introduced.
- Stability of Wilton ripple solutions to Euler's equations is examined.
- New instabilities are present due to the resonance condition being satisfied.

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ABSTRACT

Wilton ripples are a type of periodic traveling wave solution of the full water wave problem incorporating the effects of surface tension. They are characterized by a resonance phenomenon that alters the order at which the resonant harmonic mode enters in a perturbation expansion. We compute such solutions using non-perturbative numerical methods and investigate their stability by examining the spectrum of the water wave problem linearized about the resonant traveling wave. Instabilities are observed that differ from any previously found in the context of the water wave problem.

1. Introduction

In 1915 J.R. Wilton [1] included the effects of surface tension and constructed a series expansion in terms of the amplitude of one-dimensional periodic waves in water of infinite depth, extending Stokes's work [2]. He noticed that if the (non-dimensionalized) coefficient of surface tension equals $1/n$ ($n \in \mathbb{Z}^+$), the Stokes expansions giving traveling wave solutions to Euler's equations are singular. As a way to rectify the problem, he modified the form of the perturbation expansion so that the nth harmonic enters at order $(n-1)$ or $(n-2)$ instead of n. The resulting solutions are known as resonant harmonics or Wilton ripples.

The occurrence of Wilton ripples is not merely a mathematical phenomenon. Henderson and Hammack [3] generated and observed such waves in a controlled tank experiment. In the experiment, several sensors were placed at different points along the length of the tank. They measured the wave profile and the frequencies of the wave as it traveled down the tank. Even though waves of roughly 20 Hz were generated by the paddles at one end of the tank, frequencies around 10 Hz were observed as well. This is a manifestation of Wilton ripples.

McGoldrick contributed significantly to the understanding of gravity–capillary waves and their relation to resonant interaction, using both experiment and theory. He demonstrated experimentally that gravity–capillary waves lose their

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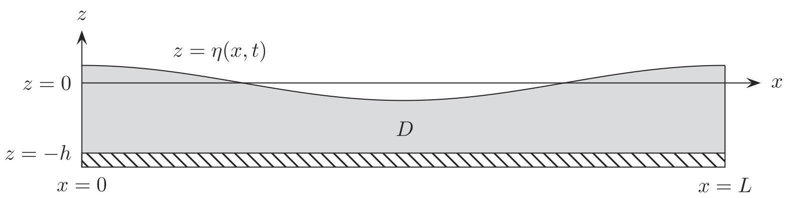

Fig. 1. The domain on which we solve Euler's equations.

initial profile as they propagate [4]. He also examined these waves using weakly nonlinear theory [5] and used the method of multiple scales [6] to investigate the evolution of the gravity–capillary waves. Further, resonant phenomena such as Wilton ripples have been studied in model equations. For instance Boyd and Haupt [7] investigated Wilton ripples in the context of the so-called super Korteweg–de Vries or Kawahara [8] equation by adding resonant harmonics into the series expansion, following Wilton's original approach [2]. Akers and Gao [9] derived an explicit series solution for the Wilton ripples in this same context.

It is known that capillary–gravity waves exhibit a Benjamin–Feir instability [10], but not much work has been done analyzing the stability of Wilton ripples outside of that. In fact, we are aware only of the work of Jones [11,12]. He investigated a system of coupled partial differential equations describing up to cubic order the interaction of the fundamental mode of a gravity–capillary wave with its second harmonic. He also provided wave train solutions of these equations. These were used to examine the stability of gravity–capillary waves as different parameters are varied. We will analyze the stability of resonant solutions by looking at the stability eigenvalue problem obtained by linearizing around a steady state solution. This was previously done by McLean [13] who built on numerical work of Longuet-Higgins [14,15] as well as others to examine growth rates of instabilities as a function of wave steepness. We will also use the ideas seen in MacKay and Saffman [16] and use the Hamiltonian structure of the problem in order to find where instabilities can occur.

In this paper, working with fully nonlinear solutions of the water wave equations, we investigate the spectral stability of resonant gravity–capillary waves using the Fourier–Floquet–Hill method [17]. To our knowledge, our work presents the first study of the different instabilities to which Wilton ripples are susceptible, without restricting the nature of the disturbances. Our paper follows the previous investigations on the instabilities of one-dimensional periodic traveling gravity waves [18] and of gravity waves in the presence of weak surface tension [19]. More details and a more comprehensive investigation of the different types of solutions, their series expansions, and their instabilities will be published elsewhere [20].

2. Computing resonant gravity–capillary waves

One-dimensional gravity–capillary waves are governed by the Euler equations,

$$\phi_{xx} + \phi_{zz} = 0, \qquad (x, z) \in D, \tag{1a}$$

$$\phi_z = 0, \qquad z = -h,\ x \in (0, L), \tag{1b}$$

$$\eta_t + \eta_x \phi_x = \phi_z, \qquad z = \eta(x, t),\ x \in (0, L), \tag{1c}$$

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = \sigma \frac{\eta_{xx}}{\left(1 + \eta_x^2\right)^{3/2}}, \qquad z = \eta(x, t),\ x \in (0, L), \tag{1d}$$

which incorporate the effects of both gravity and surface tension, where g is the acceleration due to gravity and σ is the coefficient of surface tension. Here h is the height of the fluid when at rest, $\eta(x, t)$ is the elevation of the fluid surface and $\phi(x, z, t)$ is the velocity potential. As was shown in [21], we can add an arbitrary function $C_\phi(t)$ (of time but not space) to the Bernoulli condition (1d), which we will do for computational purposes below. We focus on solutions on a periodic domain $D = \{(x, z) \mid 0 \le x < L, -h < z < \eta(x, t)\}$ as shown in Fig. 1. It is clear that the parameter space for the traveling wave solutions of this problem is extensive. A comprehensive investigation will be presented in [20]. In this brief communication, we restrict our attention to solutions for which $g = 1$, the period $L = 2\pi$ and the water depth $h = 0.05$. If one employs the criteria of [22–24], this puts us in the shallow water regime, quite different from Wilton [1] who worked with $h = \infty$. However, it should be noted that the above references distinguishing shallow water from deep water do not incorporate surface tension, and as such their results do not immediately apply.

The regular perturbation expansion (or Stokes expansion) for a 2π-periodic traveling water wave takes the form

$$\eta(x) = \epsilon \cos x + \sum_{k=2}^{\infty} \epsilon^k \eta_k(x), \quad \eta_k(x) = \sum_{j=2}^{k} 2\hat{\eta}_{kj} \cos(jx), \tag{2}$$

where the Euler equations are reduced using the traveling wave reduction $\partial_t \to -c\partial_x$. Regular perturbation theory (see, for instance, [25]) leads to an expression for $\eta_k(x)$ with a denominator proportional to the left-hand side of

$$(g + \sigma)k \tanh(h) - \left(g + k^2\sigma\right) \tanh(kh) = 0, \quad (k \neq 1). \tag{3}$$