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Integral expressions for mountain wave steepness

Harold Knight [a,*], Dave Broutman [a], Stephen D. Eckermann [b]

[a] Computational Physics, Inc, Springfield, VA, USA

[b] Space Science Division, Naval Research Laboratory, Washington DC, USA

HIGHLIGHTS

- We study the wave steepness of mountain waves, a measure of wavefield nonlinearity.
- The theory is complicated by a singularity in the Fourier integral representation.
- We focus on solutions near the ground, where the mathematics is most challenging.

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ABSTRACT

Expressions are derived for the wave steepness η_z of mountain waves, defined as the altitude (z) derivative of the vertical displacement η. The derivation begins with a known Fourier integral for η. The altitude derivative of the Fourier integral for η introduces a singularity in the integrand that is not absolutely integrable at any altitude. It is seen, nonetheless, that the altitude derivative can be moved under the integral sign for altitudes above the ground, but not at the ground, for which a special expression is found relating wave steepness to the mountain height function. It is shown that the upwind limit of wave steepness at the ground is zero, but the downwind limit is nonzero in general, and an expression for the downwind limit in terms of the mountain height function is given. It is also found that the imaginary part of complex wave steepness diverges approaching the ground, and an asymptotic formula is given. The results are applied to three mountain topography shapes: elliptical Gaussian, circular with algebraic decay, and infinite ridge.

1. Introduction

Mountain waves are internal gravity waves generated by wind flow over topography (e.g., [1,2]). Linear theory often gives a good description of mountain waves; however, as usual in wave problems it is desirable to identify regions where nonlinearity becomes important. The nonlinearity can be associated with wavebreaking above the mountain or with the effects of flow stagnation and separation near the mountain.

One of the quantities used to assess the level of nonlinearity for mountain waves (and gravity waves more generally) is the wave steepness η_z, defined as the vertical z derivative of the vertical displacement η of the mountain waves. This is related to an inverse Froude number (see Section 5.3), which is an important diagnostic quantity for stratified flow dynamics near the mountain (e.g., [1,2]). In this paper we derive integral expressions for the wave steepness in the linearized, Boussinesq, and hydrostatic approximation. We give particular attention to the behavior of the solution at the ground, which provides a chance to clarify and extend a result of Smith [3, Eq. 47], in which he gave a closed form expression for η_z at the ground for a mountain with a particular circular bell-shaped height profile.

* Corresponding author. Tel.: +1 703 764 7501; fax: +1 703 764 7500.
E-mail address: knight@cpi.com (H. Knight).

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Smith [3] derived many theoretical solutions in addition to his one formula for the wave steepness, and he provided a thorough physical interpretation of his theoretical solutions. There are numerous other references for the physics and theory of mountain waves (e.g., [1]), but very little has been written on the theoretical issues associated with the derivation of the wave steepness. For this reason, and because our derivations have subtle aspects, we concentrate mostly on the theory.

We consider both the real and complex forms for the wave steepness. (Note that Smith [3] considered only the real form.) The complex form is defined in terms of the Hilbert transform [4] and is useful because its magnitude approximates the local wave amplitude or envelope function for the wavefield. For global solutions, as we derive here, the real-valued form of the solution is sufficient for many purposes. In some applications, however, for various practical reasons, the mountain wave solutions are only calculated locally, for example on the vertical axis directly above the mountain. The amplitude or envelope function then gives an estimate of the peak value of the solution in the surrounding region, a region comparable in horizontal size to the mountain itself. One such application that typically uses solutions on a vertical axis above the mountain is the parameterization of mountain waves in weather models [5], which is one of the most important uses for mountain wave forecasting.

We find that the imaginary part of the complex solution for the wave steepness diverges to infinity at the ground. We show that the divergence is logarithmic in the altitude coordinate as the ground is approached. The divergence is a failure of the complex form to provide a suitable envelope function near the ground, and is not by itself an indication of strong nonlinearity. This type of failure is well-known in signal processing [4], where Hilbert transforms are commonly used. The source of the problem is a discontinuity in the real-valued solution at infinity between the upwind and downwind directions, as will be explained in Section 4.4.

We introduce the governing equations in Section 2, along with the Fourier integral representation of the wavefield, on which much of the theoretical work on mountain waves (e.g., [3]) is based. Next in Section 3, we define the complex form of the solution.

In Section 4 we present some new results on η_z that are obtained from the Fourier integral representation (with sketches of proofs given later in the Appendix). We establish that, despite the presence of a singularity that is not absolutely integrable, the z-derivative of η can be moved inside the Fourier integral when z is strictly above the ground. Next we clarify the mathematical relationship between η_z at the ground and the mountain height function by providing an expression that applies for general mountain height functions. We find that, at the ground, simply moving the z-derivative inside the integral for η to obtain η_z would miss part of the solution. Then we examine the complex form of η_z near the ground, finding that its imaginary part diverges, and we give an expression for its asymptotic behavior as $z \to 0$, where $z = 0$ is the ground.

In Section 5, the results of Section 4 are applied to three types of mountain height functions: elliptical Gaussian, the circular function from [3], and an infinitely long ridge. We confirm Smith's formula and examine the dependence on the horizontal aspect ratio of the mountain, which varies considerably in realistic topography. We summarize our work in Section 6.

2. Governing equations

Following Smith [3], we consider the steady flow of a vertically unbounded stratified fluid over topography of height $z = h(x, y)$. In the hydrostatic and Boussinesq approximations, the linearized governing equations of motion, continuity, and density are

$$\rho_0 U u_x = -p_x, \tag{2.1}$$

$$\rho_0 U v_x = -p_y, \tag{2.2}$$

$$0 = -p_z - \rho g, \tag{2.3}$$

$$u_x + v_y + w_z = 0, \tag{2.4}$$

$$\rho = -\rho_{0z}\eta. \tag{2.5}$$

The coordinate system is x, y, z with positive x as the downwind direction, y as the horizontal cross-wind direction, and z as the vertical direction. The background has constant wind $U > 0$ and mean density $\rho_0(z)$. The perturbation wind vector is (u, v, w), and the perturbation density and pressure are ρ and p, respectively. The vertical displacement is η. For steady linearized flow $w = U\eta_x$.

The above equations are combined to obtain

$$\partial_x^2 \partial_z^2 \eta + \frac{N^2}{U^2}(\partial_x^2 \eta + \partial_y^2 \eta) = 0, \tag{2.6}$$

where N is the buoyancy frequency, assumed constant, and defined such that

$$N^2 = -g\rho_{0z}/\rho_0. \tag{2.7}$$

We solve (2.6) by a Fourier transform method, which is a common approach in mountain wave studies (e.g., [1,6–10,3]). We express η as the Fourier integral

$$\eta(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}(k, l) \exp[i(kx + ly + mz)]\, dk\, dl \tag{2.8}$$