Wave Motion 56 (2015) 43–51

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Inverse scattering problem from an impedance crack via a composite method

Kuo-Ming Lee

Department of Mathematics, National Cheng Kung University, Taiwan

HIGHLIGHTS

- A linear regularized method which consists of two parts is proposed to recover both the impedance function and the unknown crack simultaneously from the far-field pattern with only one incident wave.
- The 2-steps scheme is very easy to implement in practice.
- No extra equations are needed other than those for the direct problem.

ARTICLE INFO

Article history:
Received 14 November 2014
Received in revised form 6 February 2015
Accepted 11 February 2015
Available online 2 March 2015

Keywords:
Inverse scattering
Impedance boundary condition
Crack

ABSTRACT

In this paper we consider an inverse scattering problem from an open arc with impedance boundary conditions on both sides of the crack. Our aim is to recover both the impedance function and the unknown crack simultaneously from the far-field pattern with only one incident wave. Making the most out of the direct problem, a straightforward method of iterative nature is developed for the inverse problem. The ill-posedness of this problem is considered by incorporating the Tikhonov regularization. Numerical examples are provided at the end of the paper to show the feasibility of our method.

1. Introduction

Crack detecting is an important issue in many areas of engineering and science. Often it is only laborious or it is dangerous to find the cracks directly. Making use of the scattering phenomenon, this task can be resolved at some convenient place distance away as an inverse scattering problem.

In this paper, we consider a more general case which is the inverse scattering problem from an impedance crack in the plane. The impedance boundary conditions can be used to model practical problems like surface coating which has its application in detection of buried objects, antenna design or the analysis of the earth surface or can be useful in the detection of the corrosion of a pipeline, the flaking or the oxidizing of a wire, for example.

From the theoretical perspective, the crack problem is more difficult than obstacle problem since the solution (of the direct problem) is not smooth in the former case. As reported in [1], the solution has a square root singularity at the end points of the arc. Utilizing the cosine substitution developed in [2], in the first article on inverse scattering problem from an open arc [3], Kress has overcome this difficulty for a sound-soft crack with an iterative Newton's method based on boundary integral equations method. This approach was extended to a Neumann crack in [4,5]. Extending to the impedance problem, although the method mentioned above did enable an elegant existence analysis, it yielded only less convergence rates in the numerical solution (see [6]). The reason for this behavior lies in the fact that in addition to the square root singularity which can be completely cleared out by the cosine substitution as in the Dirichlet or Neumann problem, the solution of

E-mail address: kmlee@mail.ncku.edu.tw.

http://dx.doi.org/10.1016/j.wavemoti.2015.02.002

the impedance problem also has a singularity of the logarithmic type at the crack tips (see also [7]) which leads to a slower convergence in the numerical computation.

Taking another approach, a linear sampling method was used to recover both the unknown crack and the impedance function in [8]. Although the reconstruction of the arc itself is rather successful, the reconstruction of the impedance is not satisfactory. A more recent work [9] based on the factorization method however, improved the quality of the reconstruction. In [10], a direct method was proposed to recover the impedance based on the analysis of [6]. However, the crack itself was considered given.

In this paper, we propose an iterative method for reconstructing the crack and the impedance simultaneously based on a system of boundary integral equations. Our method will incorporate the analysis in [6,10] neatly. The plan of the paper is as follows. In Section 2, a brief summary of the main results and the solution method for the direct problem are provided for the sake of completeness and the understanding of the inverse problem. The formulation of an inverse impedance problem and also the solution scheme are described in Section 3. This will be followed by some numerical examples in Section 4. Some conclusions and remarks will be made in the final section.

2. Direct impedance problem

The scattering of time-harmonic acoustic or electromagnetic waves from thin infinitely long cylindrical coated objects can be mathematically modeled by an impedance boundary value problem for the Helmholtz equation in the exterior domain of a crack. By a crack we mean an open arc in the plane which allows an injective and three times differentiable parametrization throughout this paper.

To be more specific, we denote the crack by Γ which is the set

$$\Gamma = \{z(t) : t \in [-1, 1]\} \subset \mathbb{R}^2$$

with an injective C^3 vector function $z : [-1, 1] \to \mathbb{R}^2$. The two end points of this open arc $z(-1), z(1)$ will be denoted by z_{-1}, z_1, respectively. We set $\Gamma_0 := \Gamma \setminus \{z_{-1}, z_1\}$. The orientation of Γ is assumed to be from z_{-1} to z_1. Furthermore we denote by Γ_+ and Γ_- the left- and right-hand sides of Γ, respectively. The unit normal vector to Γ directed towards Γ_+ is denoted by ν.

For a plane incident wave $u^i(x, d) := e^{ik\langle x,d\rangle}$ with a unit vector d indicating the direction of propagation and a wave number $k > 0$, we consider the following scattering problem which aim is to find the wave u^s scattered from the crack Γ such that the total field $u := u^i + u^s$ satisfies the homogeneous impedance boundary conditions on both sides of the crack.

Problem 1 (*The Direct Impedance Scattering Problem*). Find a solution $u^s \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\mathbb{R}^2 \setminus \Gamma_0)$ which satisfies the following conditions:

1. u^s is continuous at the two end points z_{-1}, z_1.
2. $\Delta u^s + k^2 u^s = 0$ in $\mathbb{R}^2 \setminus \Gamma$ with a wave number $k > 0$.
3. The limits
$$u^s_\pm(x) := \lim_{h\to 0^+} u^s(x \pm h\nu(x)), \quad x \in \Gamma \tag{1}$$
and the normal derivatives
$$\frac{\partial u^s_\pm(x)}{\partial \nu} := \lim_{h\to 0^+} \left\langle \nu(x), \operatorname{grad} u^s(x \pm h\nu(x))\right\rangle, \quad x \in \Gamma_0 \tag{2}$$
exist in the sense of local uniform convergence.
4. For $\lambda \in C^{0,\alpha}(\Gamma)$, $0 < \alpha < 1$, with $Re(\lambda) \geq 0$, it holds the following impedance boundary conditions
$$\frac{\partial u_\pm}{\partial \nu} \pm ik\lambda u_\pm = 0 \quad \text{on } \Gamma_0 \tag{3}$$
5. u^s satisfies the *Sommerfeld radiation condition*, i.e.
$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial \nu} - iku^s\right) = 0, \quad r := |x|,$$
uniformly in all directions $\hat{x} := \frac{x}{|x|}$.

The well-posedness of this direct impedance problem was established in [6] via boundary integral equations approach. For our purpose, we shall briefly sketch it here. In terms of the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$

$$\Phi(x, y) := \frac{i}{4} H_0^{(1)}(k|x-y|), \quad x \neq y,$$

a combination of a single layer potential and a double layer potential

$$u^s(x) := \int_\Gamma \frac{\partial \Phi(x, y)}{\partial \nu(y)} \varphi_1(y) ds(y) + \int_\Gamma \Phi(x, y)\varphi_2(y) ds(y), \tag{4}$$