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Coupled mode theory for acoustic resonators

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HIGHLIGHTS

- The method allows us to find the $\mathcal{S}$-matrix for open resonators with Neumann boundary conditions
- Transmission spectra are computed for 2D and 3D acoustic resonators.
- The method could be applied for both discrete and continuous models.
- The method represents acoustic coupled mode theory.

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ABSTRACT

We develop the effective non-Hermitian Hamiltonian approach for open systems with Neumann boundary conditions. The approach can be used for calculating the scattering matrix and the scattering function in open resonator–waveguide systems. In higher than one dimension the method represents acoustic coupled mode theory in which the scattering solution within an open resonator is found in the form of expansion over the eigenmodes of the closed resonator decoupled from the waveguides. The problem of finding the transmission spectra is reduced to solving a set of linear equations with a non-Hermitian matrix whose anti-Hermitian term accounts for coupling between the resonator eigenmodes and the scattering channels of the waveguides. Numerical applications to acoustic two-, and three-dimensional resonator–waveguide problems are considered.

1. Introduction

The approach of the effective non-Hermitian Hamiltonian [1–3] has found numerous applications in various branches of physics including atomic nuclei [4,5], chaotic billiards [6–11], tight-binding models [12–16], potential scattering [17], photonic crystals [18], etc. The objective of the present paper is to revisit the concept of the effective non-Hermitian Hamiltonian in application to open resonators with the Neumann boundary conditions. The problem of resonant scattering typically involves a resonator (which could be an atom, atomic nucleus, quantum dot, microwave or acoustic cavity *etc.*) and one, two or more scattering channels that couple the resonator to the environment. The mainstream idea is to split the full Hilbert space into subspaces: subspace B formed by the eigenfunctions of discrete spectrum localized within the scattering center, and subspace C which spans the extended eigenfunctions of the scattering channels. Therefore, the exact description of open system meets a problem of matching the wave functions of discrete and continuous spectra. In 1957 Livŝic [19] and independently Feshbach in 1958 [20] introduced the idea to project the total Hilbert space onto the discrete

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states of subspace B. Given the Hamilton operator of the whole system as

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_B + \sum_C (\widehat{\mathcal{H}}_C + V_{BC} + V_{CB}) \tag{1}$$

the projection onto the discrete subspace leads to the concept of the effective non-Hermitian Hamiltonian [1–3,20]

$$\widehat{\mathcal{H}}_{eff} = \widehat{\mathcal{H}}_B + \sum_C V_{BC} \frac{1}{E^+ - \widehat{\mathcal{H}}_C} V_{CB}. \tag{2}$$

Here $\widehat{\mathcal{H}}_B$ is the Hamiltonian of the closed system, $\widehat{\mathcal{H}}_C$ is the Hamiltonian of the scattering channel C, V_{BC}, V_{CB} stand for the coupling matrix elements between the eigenstates of $\widehat{\mathcal{H}}_B$ and the eigenstates of the scattering channel C, and E is the energy of scattered particle (wave). The term $E^+ = E + i0$ ensures that only outgoing waves will be present in the solution after the scattering occurs. As a result the effective Hamiltonian (2) is a non-Hermitian matrix with complex eigenvalues z_λ which determine the energies and lifetimes of the resonant states as $Re(z_\lambda)$, and $-2\,Im(z_\lambda)$ correspondingly [1,3]. If the propagation band of waveguide is infinite then the effective non-Hermitian Hamiltonian takes the most simple form widely used in the scattering theory [1,2,4,5,11]

$$\widehat{\mathcal{H}}_{eff} = \widehat{\mathcal{H}}_B - i\pi \sum_{C=1} W_C W_C^\dagger, \tag{3}$$

where W_C is a column matrix whose elements account for the coupling of each individual inner state to the scattering channel C, and the symbol $\dagger$ stands for Hermitian transpose. The scattering matrix $\mathcal{S}_{CC'}$ is then given by the inverse of $E - \widehat{\mathcal{H}}_{eff}$ [2,6]

$$\mathcal{S}_{C'C}(E) = \delta_{C'C} - 2\pi i W_{C'}^\dagger \frac{1}{E - \widehat{\mathcal{H}}_{eff}} W_C, \tag{4}$$

where $\delta_{C'C}$ is the Kronecker delta.

The approach of the effective non-Hermitian Hamiltonian for two dimensional resonator–waveguide systems controlled by the Schrödinger equation was previously addressed in Refs. [6,8,9]. In particular in Ref. [8] the authors derived exact formulas for the coupling matrices W_C for both Dirichlet and Neumann Boundary conditions on the boundary of the resonator. What is more interesting, however, it was shown [8] that only in the case of the Neumann boundary condition the approach is stable with respect to truncation of the discrete basis to a finite number of eigenstates thanks to the absolute convergence of the spectral sum for the reaction matrix. In the above Refs. [6,8,9] the resonator–waveguide problem was considered in the context of quantum scattering. This imposes restrictions on the applicability of the effective non-Hermitian Hamiltonian because one normally requires the Dirichlet boundary conditions on the infinitely high hard–wall boundary. Although, there are techniques to improve the convergence of the spectral sum [21,22] in the Dirichlet case, in general one would resort to the mixed boundary conditions applying the Dirichlet boundary conditions on the physical boundaries while the Neumann boundary condition is applied on the waveguide–resonator interface [8]. One the other hand, the Neumann boundary conditions are the boundary conditions for the pressure field on a sound hard boundary. That prompts us to apply the effective non-Hermitian Hamiltonian formalism to acoustic scattering problem. In essence, the proposed approach relying on the spectral properties of a closed resonator decoupled from the environment is analogous to the coupled mode theory [23,24] which is a very popular tool for analyzing resonant scattering in optics. Technically, the optical coupled mode theory [23,24] represents a method for finding transmission spectra from a set of linear equations with a matrix analogous to Eq. (3) in which the diagonal Hermitian term consists of the eigenfrequencies of the resonator, while the second anti-Hermitian term accounts for the coupling between the eigenmodes of the resonator with the scattering channels. In this paper we will focus on developing acoustic coupled mode theory including applications to discretized systems which render the method applicable to open acoustic resonators of arbitrary shape in which the eigenfunctions could not be found analytically.

The article is organized as follows. In Section 2 we consider a simple one-dimensional tight-binding model which is aimed to illustrate our approach to derive the effective non-Hermitian Hamiltonian. In Section 3 we extend our results to 2D case and demonstrate the connection between the discrete model based on the finite-difference representation of the Helmholtz equation and the continuous model based on the eigenfunctions of the partial differential equation. In Section 3 we demonstrate an application of the acoustic coupled mode theory for finding the transmission spectra and scattering functions in a realistic 3D structure. Finally, we conclude in Section 5.

2. The effective non-Hermitian Hamiltonian for one-dimensional system with the Neumann boundary conditions

There are many approaches to establish the effective non-Hermitian Hamiltonian formalism [1–3,6,12,13,17]. In this paper we adopt a variation of the method recently developed for two-particle Bose–Hubbard lattice model [25]. To describe the method we begin with the simplest possible one-dimensional tight-binding model which consists of one-dimensional resonator coupled to one or two half-infinite wires (waveguides). The systems under consideration are sketched in Fig. 1 We will see later that this approach can be easily generalized to higher dimensions as well as applied to the continuous limit which corresponds to 2D and 3D acoustic problems.