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Numerical detection and generation of solitary waves for a nonlinear wave equation

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HIGHLIGHTS

- Automatic algorithm for detecting and generating solitary waves of nonlinear wave equations.
- Dynamic simulations of the solution of a nonlinear wave equation.
- Numerical approximation.
- Improvement of the iterative cleaning technique.
- Geometric integration of solitary waves.

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ABSTRACT

This paper presents an automatic algorithm for detecting and generating solitary waves of nonlinear wave equations. With this purpose, dynamic simulations are carried out, the solution of which evolves into a main pulse along with smaller dispersive tails. The solitary waves are detected automatically by the algorithm by checking that they have constant amplitude and are symmetric respect to its maximum value. Once the main wave has been detected, the algorithm cleans the dispersive tails for time enough so that the solitary wave is obtained with the required precision.

In order to use our algorithm, we need a spatial discretization with local basis. The numerical experiments are carried out for the BBM equation discretized in space with cubic finite elements along with periodic boundary conditions. Moreover, a geometric integrator in time is used in order to obtain good approximations of the solitary waves.

1. Introduction

The purpose of this work is to develop an automatic algorithm that allows the detection and generation of solitary waves of nonlinear wave equations. This algorithm will be useful both for detecting solitary waves in dynamic simulations as for generating unknown solitary waves of nonlinear wave equations. Several techniques have been used in the literature to compute numerically solitary waves profiles for different nonlinear equations. These techniques include fixed point iterations [1], Newton-like methods [2–4], variational methods [5,6], imaginary-type evolution methods [7] and squared-operator methods [8]. These iterative methods present local convergence and need to start the iteration with a good choice for the initial profile that does not differ much from a solitary wave. When the equation of interest belongs to a parameter-dependent family of equations and for a certain value of the parameter the expressions of the solitary waves of that equation in particular are known, continuation methods can be used [9].

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There exists another kind of technique for generating solitary waves. It is based on the fact that numerical experiments suggest that, in many cases, the initial conditions will evolve with time into a single main pulse or a train of pulses along with dispersive tails (see, for example, [10–13]). This property is called 'resolution into solitary waves' and it may be proved rigorously only for a few integrable partial differential equations, for example KdV via the inverse scattering transform, [14]. In practice, this behavior will only happen when the solitary waves are stable [15–17]. Hence, an alternative to generate solitary waves is to carry out numerical simulations for time enough so that the initial profile has evolved into one or more solitary waves along with a dispersive tail. Then, the leading pulse is isolated and cleaned, that is, the remainder of the numerical solution is eliminated somehow. Typically, the chosen pulse will not be yet a solitary wave, and then, it is used as initial condition for a new simulation. The evolution with time of this new numerical experiment will probably show again a dispersive tail that will we cleaned and the new isolated wave will be considered as the initial condition of a new numerical simulation. This procedure continues iteratively, until a good approximation to a solitary wave is obtained. This is checked by observing that the shape, the velocity and the amplitude of the wave are almost constant. This technique is sometimes called iterative cleaning and is considered, for example in [10,12,13,18]. Notice that a drawback of iterative cleaning is that the speed of the main emerging wave is not known a priori. Thus, it is not possible to construct in this way a solitary wave of a desired speed or of some other desired parameter.

To carry out the numerical simulations in practice, a finite computational window is considered and periodic boundary conditions are typically used. The technique of iterative cleaning has the disadvantage that it is used in a specific way for each numerical experiment, in the following sense. The initial boundary value problem for the equation of interest is solved numerically starting with an initial condition that is close to a solitary wave. Letting the numerical solution of this particular experiment evolve with time, an instant of time T is found so that the leading pulse has separated from the remaining dispersive tails. This time T is determined in an ad hoc way, not automatically, by observing the numerical solution of a first simulation. Time T should be long enough so that the leading pulse is isolated. But, at the same time, T should be short enough so that the dispersive tails have not interacted with the main pulse because they have reached the boundary and they have entered again inside the computational window, due to the periodic boundary conditions. Moreover, the region where the numerical solution is cleaned at this time T is also chosen specifically for the numerical experiment considered. This method of ad hoc determination of the suitable time and region for the cleaning of the numerical solution is repeated several times until a good approximation to a solitary wave of the equation is found. This technique, in the way it is used in the literature, implies a lot of manual and computational work.

In view of the drawbacks of the iterative cleaning, it would be of great interest to develop a completely automatic algorithm that, based on a cleaning technique, could be able to detect and generate solitary waves in an efficient and dynamic way. This is the purpose of this paper. More precisely, the main ideas of the algorithm we propose in this work are the following. Starting with an initial profile, our algorithm will integrate numerically the corresponding initial boundary value problem, analyzing the numerical solution at each time step of the computation and determining automatically the instant of time T when the numerical solution should be cleaned for the first time. As we have already mentioned, this time should be chosen when the main pulse is isolated from the dispersive tails, but before these smaller waves, after reaching the boundary and entering again into the computational window, alter the leading wave. In this work, we assume that the solitary waves we are looking for are symmetric with respect to their maximum value, a very common property in practice. The algorithm chooses the time T when the amplitude of the main pulse is conserved and, in addition, this symmetry condition is satisfied inside a suitable interval and for a given tolerance.

After time T, the algorithm is going to clean the numerical solution, at each time step of the computation, inside a region suitably and automatically chosen by the algorithm. The cleaning technique used is based on the one proposed in [19,20], where a dynamic algorithm for long time simulations of perturbed solitary waves and collisions of traveling waves is developed. The cleaning of the numerical solution is continued until a solitary wave is obtained with the required precision. In order to determine this final time, symmetry conditions will also be used.

Although the aim of this work is more general, it is crucial for us to be able to check that our algorithm works properly and with enough precision. That is the reason why we have chosen, as a case study, the BBM equation [21]

$$u_t + u_x + uu_x - u_{txx} = 0. \tag{1}$$

For this equation, analytic expressions of its solitary waves are known

$$u(x,t) = A\,\mathrm{sech}^2(K(x-ct-L_0)), \quad A = 3(c-1),\ K = \frac{1}{2}\sqrt{1-\frac{1}{c}}, \tag{2}$$

that will allow us to determine the precision of our algorithm. Moreover, the BBM equation has been widely used as a model for the simulation of solitary waves. We remark that the algorithm we propose is not going to use these analytic expressions and the technique is suitable for detecting and generating solitary waves of more general nonlinear wave equations for which explicit expressions are not known.

For the discretization of the initial boundary value problem for the BBM equation, we have considered a method of lines. The spatial discretization must be chosen so that the numerical solution belongs to a discrete space with local basis. In this way, the cleaning technique does not affect the numerical solution in the whole interval. For instance, finite elements may be used. More precisely, we consider, as shape functions, piecewise cubic Hermite polynomial functions for which