Wave Motion 51 (2014) 445–465

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Numerical interaction of boundary waves with perfectly matched layers in two space dimensional elastic waveguides

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HIGHLIGHTS

- We analyze the stability and accuracy of the PML in transient elastic waveguides bounded by free-surface boundary conditions.
- Using the SBP-SAT methodology we develop a stable numerical approximation of the PML in waveguides.
- We demonstrate that back-propagating modes are not harmful to a finite width discrete PML.
- Numerical experiments verify the theoretical results.

ARTICLE INFO

Article history:
Received 11 April 2012
Received in revised form 28 October 2013
Accepted 5 November 2013
Available online 22 November 2013

Keywords:
Guided waves
Perfectly matched layer
Group velocity
Stability
SBP-SAT
High order accuracy

ABSTRACT

Perfectly matched layers (PMLs) are now a standard approach to simulate the absorption of waves in open domains. Wave propagation in elastic waveguides has the possibility to support back-propagating modes (propagating modes with oppositely directed group and phase velocities) with long wavelengths. Back-propagating modes can lead to temporally growing solutions in the PML. In this paper, we demonstrate that back-propagating modes in a two space dimensional isotropic elastic waveguide are not harmful to a discrete and finite width PML. Analysis and numerical experiments confirm the accuracy and stability of the PML.

1. Introduction

The study of wave propagation in guided structures is important in many engineering applications. For instance, industrial materials such as steel plates and pipes, and natural structures like the Earth's surface, can support propagating waves that can lead to failure or disaster. Both theoretical and experimental investigations of wave propagations in guided structures can provide valuable information which can be useful in developing modern technologies or understanding natural disasters such as earthquakes and tsunamis. Numerical simulations can serve as a complement to both theoretical and experimental investigations, and can possibly treat many more important scenarios that cannot be accessed theoretically or experimentally. Wave propagation problems are often formulated in infinite or very large spatial domains. In order to perform numerical simulations, large spatial domains must be replaced by smaller computational

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http://dx.doi.org/10.1016/j.wavemoti.2013.11.002

domains, thereby introducing artificial boundaries. To ensure the accuracy of numerical simulations, artificial boundary conditions must be imposed such that all outgoing waves disappear without reflection.

The perfectly matched layer (PML) [1–5] is an effective method to simulate the absorption of waves in numerical wave simulators. For time-dependent problems, it is important that the PML ensures that all solutions remain bounded for sufficiently long times. A useful but negative stability result was established in [2]. For hyperbolic systems, it was found that if a physical model supports back-propagating modes (modes with group velocity and phase velocity of opposite signs), then at sufficiently high frequencies the corresponding PML will support exponentially growing solutions. This undesirable behavior of the PML is largely found in many anisotropic wave equations [1,2,4]. For isotropic elastic wave equations, the PML for the corresponding Cauchy problem is stable. However, boundary conditions can introduce many more difficulties. For instance in isotropic elastic waveguides, back-propagating modes can be supported [6] when free-surface boundary conditions are imposed. Second, in the discrete setting if boundary conditions are not well treated, the PML can pollute the numerical solutions everywhere [7,8,6,9].

In [8], we considered the PML in a waveguide governed by the time-dependent scalar wave equation. For a waveguide bounded by Neumann boundary conditions, to ensure the accuracy and the stability of the discrete PML, we derived a set of equivalent transformed boundary conditions. The stability of the perfectly matched layer, in the frequency domain, for isotropic elastic waveguides bounded by free-surface boundary conditions was considered in [6]. Then in a recent work [10], we considered a corresponding half-plane problem in the time domain. We derived a PML, together with two mathematically equivalent transformed boundary conditions in the PML. In order to demonstrate the stability of the PML on the half-plane, we showed that all roots of the perturbed Rayleigh dispersion relation corresponding to the PML are in the stable half-plane.

In this paper, we consider the PML in a two space dimensional transient isotropic elastic waveguide. The waveguide is bounded in the y-direction (on the top and bottom) by free-surface boundary conditions, but extends infinitely in the x-direction. Our first objective is to extend the analysis of the Cauchy problem [2] and the half-plane problem [10] to the rectangular waveguide. The corresponding dispersion relation is a non-trivial expression known as the Rayleigh–Lamb dispersion relation. By examining solutions of this relation it is clear that isotropic elastic waveguides can support back-propagating modes. We show that at sufficiently small damping the existence of back-propagating modes in the waveguide can lead to temporally growing solutions in the continuous PML. However, at sufficiently high frequencies we regain the Rayleigh dispersion relation. Therefore, unlike the Cauchy problem for anisotropic systems [1,2,4,11], there are no growing modes in the PML at high frequencies. Note that the frequency bands where back-propagating modes exist are rather small and most prominent at lower spatial frequencies [6,12]. In experiments, experimentalists often use frequencies that do not excite them, while in computations all possible modes can be excited. In practice the PML is a finite width layer and can only support modes of sufficiently short wavelengths. In many cases all back-propagating modes have longer wave lengths than can be supported in the PML, and the PML is thus a time stable system.

The second objective of this paper is to develop a stable numerical method for the PML and numerically evaluate, in a waveguide, the transformed boundary conditions proposed in [10]. We will use summation-by-parts (SBP) finite difference operators [13,14] to approximate spatial derivatives. The transformed boundary conditions are imposed weakly using the simultaneous approximation term (SAT) method [15]. We present an energy estimate for a simplified constant coefficient PML, where only variations in the span-wise direction are taken into account. In a 1D setting we derive a discretization which satisfies a similar estimate. The estimate relies on a particular formulation of the transformed free-surface boundary condition. The discretization can straightforwardly be extended to 2D, but we have not been able to extend the estimates to 2D. However, the 1D analysis guides the choice of the penalty parameter also for the 2D case.

To further investigate the stability of the 2D discretization we present eigenvalue results for a semi-discrete constant coefficient PML. These show that as long as we restrict the width of the PML so that back-propagating modes predicted by the undamped dispersion relation cannot be supported in the continuous setting, the discretization does not support growing solutions. This result relies heavily on using the particular transformation of the free-surface boundary condition which allows for a discrete energy estimate in 1D. We also investigate another formulation of the boundary condition, which is of the same accuracy, but does not yield an energy estimate in 1D. In this case the 2D eigenvalue analysis shows that other discrete growing modes than those related to back propagation appear. It would be interesting to see if corresponding results can be obtained for numerical methods derived from a variational formulation such as finite element methods.

The paper will proceed as follows. In the next section, we introduce the elastic wave equations and present the well-known results of time-harmonic wave propagation in elastic waveguides. The PML and the corresponding transformed boundary conditions are presented in Section 3. In Section 4, we present the stability analysis of the PML. Numerical approximations and discrete stability analysis are presented in Section 5. In Section 6 we perform numerical experiments that verify our analysis. In the last section we draw conclusions.

2. Linear elastodynamics

Consider the elastic wave equation for a homogeneous isotropic elastic medium in two space dimensions

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\partial}{\partial x}\left(A \frac{\partial \mathbf{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(B \frac{\partial \mathbf{u}}{\partial y}\right) + \frac{\partial}{\partial x}\left(C \frac{\partial \mathbf{u}}{\partial y}\right) + \frac{\partial}{\partial y}\left(C^T \frac{\partial \mathbf{u}}{\partial x}\right), \tag{1}$$