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Oscillatory line source for water waves in shear flow

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HIGHLIGHTS

- Water-wave radiation on a shear flow is represented by a Green function: the 2D oscillating source.
- Zero surface velocity gives no Doppler effects and at most two radiated waves.
- There exists an upstream wave with fluid particles co-rotating with the vorticity.
- The downstream wave has greater amplitude but a cut-off: it vanishes for angular frequencies below the vorticity.

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ABSTRACT

The linearized water-wave radiation problem for the oscillating 2D submerged source in an inviscid shear flow with a free surface is investigated analytically. The vorticity is uniform, with zero velocity at the free surface. Then there will be at most two emitted waves, and no Doppler effects. Exact far-field waves are derived, with radiation conditions applied at infinity. An upstream wave will always exist, whereas the downstream wave exists only when the angular frequency of oscillation exceeds the vorticity. The wave radiation problem is solved also for oscillating vortex and dipoles. The amplitudes and energy fluxes are calculated.

1. Introduction

The submerged oscillatory source is an important tool for solving linearized water-wave problems; see [1,2]. Oscillatory sources are Green functions that satisfy the linearized free-surface condition and the radiation conditions at infinity. The first mathematical solutions were given by Kochin [3]. The classical theory is summarized by Wehausen and Laitone [4]. Finite constant depth as well as a uniform basic flow has been included, but the fluid flow is restricted to be irrotational.

In the present work we study the submerged oscillatory source for the simplest possible case of a basic flow with vorticity: a two-dimensional source in a flow with uniform vorticity, where the fluid depth is infinite and the shear flow has zero velocity at the free surface.

Wehausen and Laitone [4] outlined the general problem of a time-dependent submerged source in uniform flow. Chen and Wu [5] discussed the complicated wave field due to an oscillating 3D source close to the free surface. The 2D oscillatory source is simpler, but it may emit up to four different waves, due to the Doppler effect. The wave amplitude becomes infinite according to linear theory at resonance when the group velocity is zero. Tyvand and Torheim [6] solved the radiation problem for a 2D bottom source at constant depth with uniform flow, confirming the infinite amplitude of linearized resonance for a single oscillatory source.

There are ways of avoiding infinite amplitude at resonance, even for strictly oscillatory flow. Grue and Palm [7] showed that a source distribution due to a two-dimensional oscillating body in a uniform flow will radiate waves of finite amplitude

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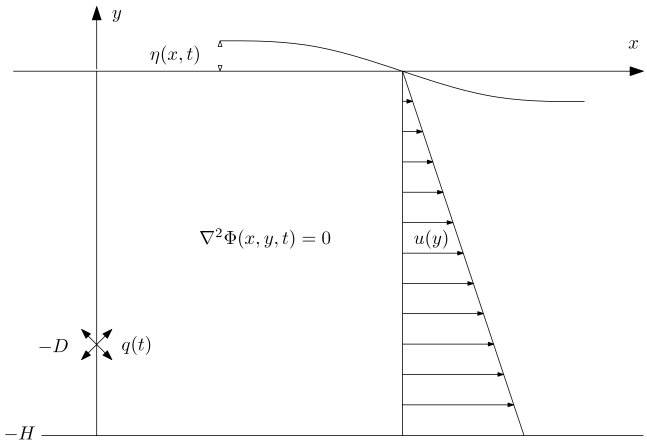

Fig. 1. A sketch of the problem at hand.

even at resonance, according to linear theory. Dagan and Miloh [8] demonstrated that nonlinear effects can make the wave amplitude from a single source in uniform flow finite at resonance.

The present work introduces submerged oscillating 2D singularities in a shear flow. We will assume zero velocity at the free surface, which is an assumption that will simplify the physics, because all Doppler effects are avoided. Then there is no resonance, and there are at most two emitted waves, with finite amplitudes. The mathematics of rotational flows is greatly simplified when the vorticity is uniform, since the individual conservation of vorticity (Helmholtz's theorem) is automatically satisfied.

Water waves on shear flows are of interest in oceanography and coastal engineering; see the review article by Peregrine [9]. Shear flows are involved in the generation of water waves by wind stress [10]. The particle motion in such wind-driven waves will co-rotate with the vorticity. Also in roll waves for viscous film flow along a slope, each wave crest will naturally perform a rotational motion in its direction of propagation [11]. In the present problem, one may speculate that the waves that are easiest to generate are those that have particle motion co-rotating with the vorticity.

2. Mathematical model

We consider an inviscid and incompressible fluid in a steady shear flow along a horizontal x axis. The flow is two-dimensional in the x, y plane, and it is driven by a fixed oscillating line source located at a depth D. The semi-infinite fluid has a free surface subject to constant atmospheric pressure. Cartesian coordinates x, y are introduced, where the y axis is directed upward in the gravity field and $y = 0$ represents the undisturbed free surface. The gravitational acceleration is g, and ρ denotes the constant fluid density. The surface elevation is denoted by $\eta(x, t)$, as can be seen in the sketch of the overall problem in Fig. 1. We assume a constant fluid depth H, even though the detailed solution will be given only for the limit $H \to \infty$. There is a basic horizontal shear flow $U(y)$ in the $+x$ direction

$$U(y) = -\Omega y, \quad -H < y < 0, \tag{1}$$

with uniform vorticity Ω, which is the velocity gradient of the steady shear flow. We consider a shear flow that leaves the undisturbed free surface $y = 0$ at rest.

The total velocity field $\mathbf{v}$ is the sum of an unsteady potential flow plus the steady shear flow

$$\mathbf{v} = \nabla\Phi - \Omega y\mathbf{i}, \tag{2}$$

where $\Phi(x, y, t)$ is the velocity potential, and $\mathbf{i}$ is the horizontal unit vector.

Conservation of mass gives Laplace's equation

$$\nabla^2\Phi = 0, \tag{3}$$

valid in the fluid outside the source point $(x, y) = (0, -D)$.

Euler's equation of motion can be written as

$$\nabla\left(\frac{p}{\rho} + \frac{\partial\Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2 + gy\right) = \Omega\left(y\nabla\frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\mathbf{i}\right). \tag{4}$$

We recognize the terms from the Bernoulli equation for irrotational flow. There are two extra terms due to vorticity.

The linearized kinematic free-surface condition is

$$\frac{\partial\eta}{\partial t} = \frac{\partial\Phi}{\partial y}, \quad y = 0. \tag{5}$$