Wave Motion 51 (2014) 533–546

Contents lists available at ScienceDirect

Wave Motion

journal homepage: www.elsevier.com/locate/wavemoti

Trapped modes due to narrow cracks in thin simply-supported elastic plates

R. Porter *, D.V. Evans

School of Mathematics, University of Bristol, Bristol, BS8 1TW, UK

ARTICLE INFO

Article history:
Received 25 April 2013
Received in revised form 23 December 2013
Accepted 1 January 2014
Available online 15 January 2014

Keywords:
Trapped modes
Thin elastic plates
Thin cracks
Wiener–Hopf method
Integro-differential equations

ABSTRACT

We consider an elastic plate of infinite length and constant width supported simply along its two parallel edges and having a finite length crack along its centreline. In particular, we look for and find trapped modes (localised oscillations) in the presence of the crack. An explicit wide-spacing approximation based on the Wiener–Hopf technique applied to incident wave scattering by semi-infinite cracks is complemented by an exact formulation of the problem in the form of integro-differential equations. An application of a Galerkin method for the numerical calculation of results from the latter method leads to a novel explicit 'small-spacing' approximation. In combination with the wide-spacing results this is shown to provide accurate results for all lengths of crack.

1. Introduction

In a recent paper, Porter [1] provided numerical evidence for localised time-harmonic out-of-plane oscillations in a thin elastic plate of infinite extent but constant width and simply supported along its two parallel edges. The mechanism by which these undamped bending waves remain trapped is provided by the presence of a circular hole cut out of the centreline of the strip. Crucially, use is made of the fact that, below a certain non-zero critical frequency, free waves are prohibited from propagating along the strip to infinity. Thus it was shown in [1] that trapped modes with displacements both symmetric and antisymmetric about the centreline of the strip are possible when the edge of the circular hole is free and for any radius provided that the circular hole is contained within the strip. In contrast no such modes are found when the edge of the circular hole is clamped.

The motivation behind the idea of Porter [1] came from earlier studies into the trapping of acoustic waves by sound-hard circles in parallel-walled acoustic waveguides [2] in which similar critical frequencies could be generated by considering motions with particular symmetries. Indeed, the method employed by Porter [1] of using multipole expansions followed methods pioneered in the work of Callan et al. [2].

A large body of work on the subject of trapped waves in acoustics (whose formulation also applies to other physical settings such as surface waves on water, quantum wires, and electromagnetics) in parallel-walled waveguide domains emerged around the same time as the work of Callan et al. [2]; see [3] for a comprehensive review. This included the work of [4] who gave a constructive proof of trapped modes for a waveguide containing a sufficiently-long rigid plate along the centreline. Earlier, Evans and Linton [5] had used numerical methods in considering rectangular obstacles in waveguides with the thin plate being a limiting case. Thus it was demonstrated that trapped waves exist for a variety of obstacles in waveguides and, in 1994, Evans et al. [6] proved the existence of trapped waves for any symmetric sound-hard obstacle placed in an acoustically-hard walled waveguide, including the geometry considered by Evans [4].

* Corresponding author. Tel.: +44 117 9287996; fax: +44 117 9287999.
E-mail addresses: Richard.Porter@bristol.ac.uk, richard.porter@bris.ac.uk (R. Porter).

0165-2125/$ – see front matter
http://dx.doi.org/10.1016/j.wavemoti.2014.01.002

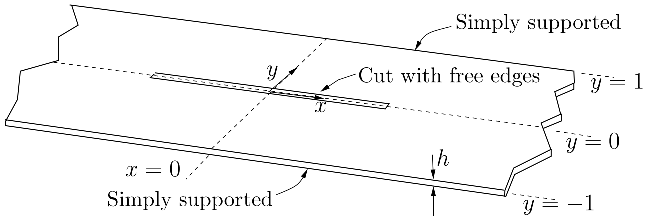

Fig. 1. Geometry of the problem.

Given this background, it would be very surprising if circular holes with free edges, as considered by Porter [1], were the only geometrical configuration capable of supporting trapped modes in thin elastic plate waveguides. Thus here we consider a geometric configuration in common with that considered by Evans [4], whereby a finite length cut — or crack — is placed along the centreline of the waveguide. Our primary aim is to provide strong evidence for the existence of trapped modes for such a configuration. A secondary purpose is to demonstrate analytical methods for doing this effectively.

Methods for solving problems involving thin cracks in elastic plates in a variety of settings have been considered by numerous authors. For example, Norris and Wang [7] sought the solution to incident plane wave scattering by a semi-infinite straight-line crack in an unbounded elastic plate using the Wiener–Hopf technique. Around the same time, Andronov and Belinskii [8] used Fourier transform methods to develop integro-differential equations for wave scattering by straight cuts of finite length, again spatially in unbounded plates. This was later extended to multiple finite length cracks in a slightly more complicated model involving elastic plates bounded below by an incompressible fluid of finite depth (a model for cracks in ice sheets) by Porter and Evans [9]. Like [8], they used a Fourier transform approach to develop integro-differential equations. They also used an expansion of the unknowns in those equations suggested by Andronov and Belinskii [8] to reduce the problem to the solution of a rapidly convergent infinite system of equations.

The approach taken in this paper is two-fold. First we develop a so-called wide-spacing approximation in which the canonical problem of the scattering of waves incident from infinity along a waveguide containing a crack of semi-infinite extent is considered. It is shown in Section 3 using the Wiener–Hopf technique, similar to that described briefly in [5], that, for frequencies below the critical first cut-off frequency of the waveguide, waves are totally reflected and simple explicit expressions are given for the phase of the complex reflected wave amplitude. This is used to provide estimates of trapped mode configurations on the assumption that the crack is sufficiently long. A second approach, outlined in Section 4 is, in common with the Wiener–Hopf method, based on Fourier transforms and results in integro-differential equations. These are converted to infinite systems of algebraic equations using the same Galerkin approach used in [9]. This motivates a novel explicit 'small-spacing' approximation to complement the wide-spacing approximation of the earlier section based upon a one-term truncation of the infinite system of equations. In Section 5 it is shown how both types of approximations compare with accurate results from a converged truncation of the system of equations based on the integral equation formulation. It is demonstrated that both approximations work well beyond the assumed limits of applicability.

Finally in Section 6 we summarise the work and discuss features of the solution method, such as not requiring knowledge of the roots of dispersion equations, that make the approaches used here attractive and adaptable to other problems.

2. Formulation of the problem

We consider an infinitely-long rectangular strip, $-1 < y < 1$, $-\infty < x < \infty$ occupied by a thin elastic plate of thickness h whose time-harmonic vibrations described by the function $\Re\{u(x, y)e^{-i\omega t}\}$ are perpendicular to the plane it occupies in equilibrium (See Fig. 1.). In the strip

$$(\Delta^2 - k^4)u = 0 \tag{1}$$

is satisfied by $u(x, y)$ where $\Delta = \partial_{xx} + \partial_{yy}$ and $k = \rho h\omega^2/D$ in terms of ρ, the areal density of the plate and D, the flexural rigidity defined as $\frac{1}{12}Eh^3/(1-\nu)$ in terms of Young's modulus E and the Poisson ratio ν. All lengths are scaled by the fixed width 2 of the plate.

On the lateral boundaries of the strip the elastic plate is simply (roller) supported, so that

$$u = 0, \quad \text{and} \quad u_{yy} = 0, \quad \text{on } |y| = 1, -\infty < x < \infty. \tag{2}$$

Along the centreline, the plate is cut on $x \in \mathcal{C}$, where $\mathcal{C} = \{x : |x| < a\}$ so that here

$$\left.\begin{aligned} (\mathcal{B}u)(x) &\equiv u_{yy}(x,0) + \nu u_{xx}(x,0) = 0, \\ (\mathcal{S}u)(x) &\equiv u_{yyy}(x,0) + (2-\nu)u_{xxy}(x,0) = 0, \end{aligned}\right\} \quad x \in \mathcal{C} \tag{3}$$

representing the vanishing of bending moment and shear stress on the free edges.

The geometric symmetry about $y = 0$ allows us to consider solutions of (1)–(3) which are symmetric/antisymmetric about $y = 0$. We denote such solutions by $u^{s/a}(x, y)$ such that

$$u^s(x, y) = u^s(x, -y) \tag{4}$$