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# On the nonlinear interpretation of a vibrating wire viscometer operated at large amplitude

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ABSTRACT

A non-linear interpretation is proposed for the vibrating wire (VW) viscometer operated with the transient or ringdown method. This new interpretation is motivated by the necessity of having an acceptable signal to noise ratio (SNR) such that a measurement can be performed in 1 s. A large SNR is achieved by a large oscillation amplitude such that the requisite conditions for the linear interpretation are no longer met. We demonstrate the applicability of a new non-linear interpretation with wires of length 40 mm and 50 mm over a viscosity range from 0.3 to 159 mPa s where the maximum amplitude of motion is approximately two thirds of the radius.



## 1. Introduction

The long history of the vibrating wire (VW) viscometer commences with Stokes's original calculation of the drag experienced by a cylinder oscillating normal to its axis [1]. More recently a full solution for the VW was derived by Wakeham and coworkers for low amplitude operation and the interpretation discussed within has been implemented in multiple studies concerning the viscosity of hydrocarbon mixtures [2,3]. This interpretation postulates that the drag experienced by the wire is linearly dependent on the wire's velocity and will be referred to as the *linear interpretation*. This condition is met for a wire oscillating with small amplitude (significantly less than its radius) and frequencies sufficiently small that convective terms are negligible [2,3]. The *emf* voltage signal produced by the VW can be measured with the steady state mode using a lock-in amplifier wherein one viscosity measurement requires a frequency sweep about the resonance peak, taking approximately 1 min. The noise-rejecting capability of the lock-in amplifier allows nanovolt signals to be measured with accuracy such that the VW can be operated at low amplitude. The transient method using an impulse excitation and voltage recorder cannot reject off-resonant frequencies as robustly as the lock-in amplifier and as such is a measurement with a higher level of background noise. Hence the transient method requires that the VW oscillate with a much higher physical amplitude in order to obtain the same SNR as the steady state approach. In the literature it is typically reported that a series of measurements with sequentially decreasing excitation is employed and the measurement of the decrement $\Delta$ by a transient method is extrapolated to zero amplitude, requiring about 15 min [4] The accuracy of the interpretation increases as the amplitude is decreased because the approximations intrinsic to the interpretation possess greater validity at low amplitude. To our knowledge, no VW viscometer has been fabricated for commercial implementation, and as such, there has been little need to perform measurements rapidly [4]. For many applications a measurement that requires even 1 min is unacceptably long, especially to a sensor subjected to heterogeneous flow consisting of slugs of different fluids.

In this manuscript we will discuss an extension of existing interpretation that allows one to make a measurement in the span of 1 s using the transient method. This builds upon the existing interpretation found in the literature but includes *non-linear terms* to

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compensate for a well-known non-linearity which arises when the VW is driven at high amplitude. This high amplitude provides a satisfactory SNR such that the measurement can be carried out rapidly, accurately, and precisely. In what follows we will first discuss the theoretical basis of this interpretation, then we will demonstrate that this non-linearity exists regardless whether the steady state or transient methods are implemented. A comparison of viscosity measurements with the non-linear and the linear interpretations is made for low viscosity fluids, and finally, a series of measurements over a wide range of viscosities (0.3–159 mPa s) is presented in the form of a discrepancy plot. The discrepancy between measured and literature values increases with viscosity and the largest discrepancy was to be −3.6% at 159 mPa s. The standard deviation for viscosities less than 50 mPa s was around 1%, which increased to 2% at 159 mPa s.

## 2. Theory

### 2.1. Damped harmonic oscillator

The amplitude of motion of a vibrating wire, $y(z)$, of circular cross-section clamped at both ends is described by the equation below and was discussed by Retsina et al. [2]. Here $y$ corresponds to the displacement normal to the wire axis, the latter being oriented along the $z$ direction.

$$EIy_{zzzz} - Ty_{zz} + (m_0 + m_f)\ddot{y} + (D_0 + D_f)\dot{y} = F_f(z, t) \quad (1)$$

The components of this equation can be classified into either mechanical or fluid properties. The mechanical properties consist of the Young's modulus $E$, axial tension $T$, moment of inertia $I$, rod mass per unit length $m_0$, and internal damping per unit length $D_0$. The fluid properties consist of the added fluid mass per unit length $m_f$ and the fluid drag per unit length $D_f$. The right hand term is the drag associated with flow startup $F_f(z,t)$, which we will neglect in this discussion. We are interested in understanding the effects of fluid mechanics in this system, not on the detailed mechanics of the rod, so we proceed to separate the solid mechanics problem from the fluid mechanics one. To do this, we make the assumption that axial flow in our system is negligible. This assumption is equivalent to treating the system as a collection of two-dimensional flow problems, each identical except for the amplitude of oscillation. If we make the further assumption that the fluid mechanical parameters $m_f$ and $D_f$ are not spatially dependent, this allows the fluid and solid mechanics to be decoupled. Using separation of variables we can isolate the temporal portion of the Eq. (1) and add a single stiffness constant, which we will term $k$. Rewriting Eq. (1) we obtain:

$$m\ddot{y} + D_L\dot{y} + ky = 0 \quad (2)$$

where $m = m_0 + m_f$ and $D_L = D_0 + D_f$. This can be recognized as the equation of a damped harmonic oscillator and its solutions will be the foundation of our analyses. The full case can also be treated, but the additional complications are not instructive as to the general method and are only presented in Appendix A. The linear drag, $D_L$, and total mass $m$, will in general depend upon the fluid viscosity, fluid density, wire dimensions, resonant frequency, and vibrational amplitude. For small amplitudes, however, both the drag and total mass will be constants that do not depend on amplitude. In such a case, the motion will have a particularly simple solution $y_0$ given by

$$y_0 = \exp(-\Delta\omega t)(C_{0,1}\ \exp\iota\omega t + C_{0,0}\exp - \iota\omega t) \quad (3)$$

where $\omega^2 = k/m - (D_L/2m)^2$ and $\Delta\omega = D_L/2m$. The coefficients $C_{0,0}$ and $C_{0,1}$ are determined by the initial conditions of the motion, with the caveat that the real motion comes from the real part of Eq. (3). For comparison to experiment, it is convenient to rewrite Eq. (3) as

$$y_0 = V_0\exp(-\Delta\omega t)\cos(\omega t + \phi_0) \quad (4)$$

where $\phi$ is the phase shift.

### 2.2. Weakly nonlinear damping

If the amplitude of vibration becomes large, the drag and added mass coefficients are no longer constant, but vary with amplitude. This condition can be met by either increasing the wire amplitude for a given viscosity or decreasing the fluid viscosity for a given amplitude. We can define the non-dimensional amplitude $\varepsilon$ as $y/R$, where again $R$ is the wire radius and the Reynolds number is defined as the $\rho\omega yR/\eta$. The amplitude is typically considered small where $\varepsilon$ is significantly less than 0.1 and our experimental results will help us to determine where the viscosity and hence Reynolds number plays a role. We can expand the drag in a power series of velocity to produce the nonlinear drag $D_{NL}$.

$$D_{NL} = D_L(1 + \alpha_1\dot{y}^2 + \alpha_2\dot{y}^4 + \ldots) \quad (5)$$

The odd power terms are eliminated as they produce unphysical drag. A similar expansion could be performed for added mass, or with alternate forms for the drag expansion, but such discussion is left to Appendix B. The parameters $\alpha_i$ describe the power series of the fully nonlinear drag. Keeping only the smallest term in the expansion of drag, the oscillator equation becomes

$$m\ddot{y} + D_L(1 + \alpha_1\dot{y}^2)\dot{y} + ky = 0 \quad (6)$$

We solve this equation perturbatively, starting from the linear solution $y_0$ where $\alpha_1 = 0$. The total solution $y$ can be rewritten as $y = y_0 + y_1$, which can be used to define the difference $y_1$, where $y_1 \ll y_0$. Substituting this first-order definition of $y$ into Eq. (6) and subtracting the linear result ($m\ddot{y}_0 + D_L\dot{y}_0 + ky_0 = 0$) we obtain:

$$m\ddot{y}_1 + D_L\dot{y}_1 + ky_1 = -D_L\alpha_1(\dot{y}_0)^3 - D_L\alpha_1(3y_1y_0^2 + 3y_1^2y_0 + y_1^3) \quad (7)$$

By eliminating terms on the right hand side that include factors of the small perturbation $y_1$ we can rewrite the above equation as:

$$m\ddot{y}_1 + D_L\dot{y}_1 + ky_1 = -D_L\alpha_1\dot{y}_0^3 \quad (8)$$

This can then be generalized to solve for all subsequent $y_n$ as the solution to the linear equation for $y_n$ and all remaining nonlinear terms less than $n$.

Using our previous solution for $y_0$ and Eq. (8) yields a solution for $y_1$

$$y_1 = \exp(-3\Delta\omega t)\left(\sum_{n=0}^{n=3} C_{1,n}\ \exp((n-3)\iota\omega t)\right) \quad (9)$$

where the coefficients $C_{1,n}$ are related to the coefficients $C_{0,p}$ by

$$C_{1,n} = \frac{\binom{3}{n} D_L\alpha_1[C_{0,0}(-\Delta\omega - \iota\omega)]^{2n-3}[C_{0,1}(-\Delta\omega + \iota\omega)]^{3-2n}}{m(-3\Delta\omega + (2n-3)\iota\omega)^2 + D_L(-3\Delta\omega + (2n-3)\iota\omega) + k} \quad (10)$$

The first index of $C$ corresponds to the order of the expansion and the second index specifies the coefficient term. In more experimentally practical notation, this becomes

$$y_1 = \exp(-3\Delta\omega t)(V_{1,3}\cos(3\omega t + \phi_{1,3}) + V_{1,1}\cos(\omega t + \phi_{1,1})) \quad (11)$$