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Review

# Solvers for the cardiac bidomain equations

E.J. Vigmond[a,*], R. Weber dos Santos[b], A.J. Prassl[c], M. Deo[a], G. Plank[c]

[a]*Department of Electrical and Computer Engineering, University of Calgary, Calgary, Alta., Canada*
[b]*Department of Computer Science and Program on Computational Modeling, Universidade Federal de Juiz de Fora, Juiz de Fora, Brazil*
[c]*Institute of Biophysics, Center for Physiological Medicine, Medical University of Graz, Graz, Austria*

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**Abstract**

The bidomain equations are widely used for the simulation of electrical activity in cardiac tissue. They are especially important for accurately modeling extracellular stimulation, as evidenced by their prediction of virtual electrode polarization before experimental verification. However, solution of the equations is computationally expensive due to the fine spatial and temporal discretization needed. This limits the size and duration of the problem which can be modeled. Regardless of the specific form into which they are cast, the computational bottleneck becomes the repeated solution of a large, linear system. The purpose of this review is to give an overview of the equations and the methods by which they have been solved. Of particular note are recent developments in multigrid methods, which have proven to be the most efficient.


*Keywords:* Cardiac electrical modeling; Bidomain model; Multigrid; Preconditioning

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*Corresponding author. Tel.: +1 403 802 1577; fax: +1 403 282 6855.
*E-mail address:* vigmond@ucalgary.ca (E.J. Vigmond).

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## 1. Introduction

The set of bidomain equations (Sepulveda et al., 1989) is currently the most complete mathematical models for describing the spread of cardiac electrical activity, especially for simulating activity on the organ level. They were first applied to cardiac tissue in 1978 (Miller and Geselowitz, 1978; Tung, 1978). The system of non-linear partial differential equations (PDEs) models both the intracellular and extracellular domains of the cardiac tissue from an electrostatic point of view. The coupling of the two domains is performed via non-linear models describing the current flow through the cell membrane, which leaves one domain to enter the other. Recently, validation of the bidomain model through animal experiments (Wikswo et al., 1995; Muzikant and Henriquez, 1998) as well as its ability to reproduce cardiac phenomena (see Lin and Wikswo, 2001; Roth, 2001; Trayanova et al., 1998; Wikswo, 1994; Trayanova, 2006) makes the bidomain model a strong candidate for simulating cardiac electrical behavior. In fact, the bidomain equations predicted the existence of virtual electrodes (Sepulveda et al., 1989), which were later experimentally validated (Wikswo et al., 1995), and have become instrumental in understanding defibrillation (Trayanova, 2006). See Fig. 1 for an example of the complex polarization patterns. When bath loading effects, extracellular stimulation, or the magnetic field is to be modeled (Weber dos Santos et al., 2003; Weber dos Santos and Dickstein, 2003) accurately, it is the only choice.

Several factors contribute to make the bidomain simulations inherently computationally burdensome. Spatially, a fine discretization is needed to properly capture field effects. When applying an extracellular shock in tissue, its directly elicited response decays with a space factor on the order of hundreds of micrometers.

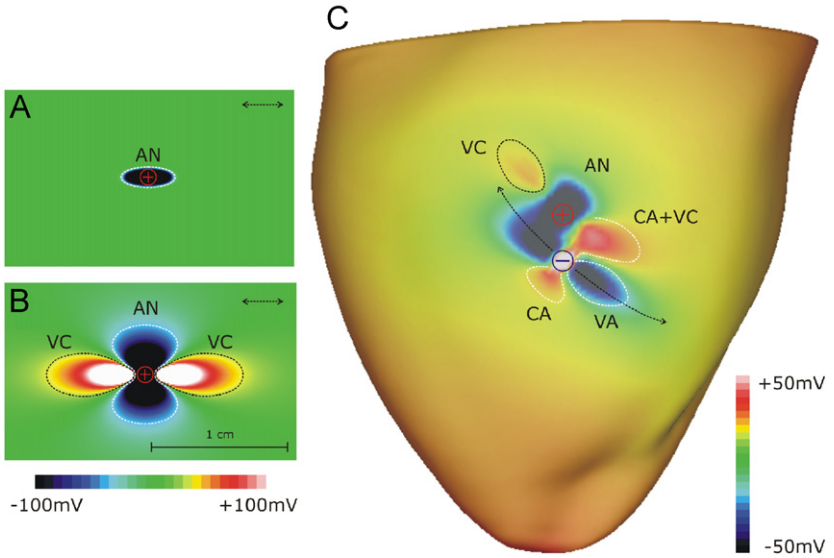


Fig. 1. Transmembrane polarization induced by extracellular stimulation. (A) Unipolar anodal stimulation of a thin 3D sheet with equal anisotropy ratios. Only changes in transmembrane voltage of one polarity (anodal hyperpolarization in this case) are observed. These decay quickly within a few space constants from the stimulation site, with the space constant being a function of fiber orientation (indicated by the arrow). The location of the stimulus is indicated by a battery pole. The ellipsoidal region of stimulus-induced hyperpolarization under the positive pole is referred to as the anode (AN). (B) With unequal anisotropy ratios, the polarization pattern is strikingly different. Changes in transmembrane voltage of both polarities are observed. Compared to (A), the region of anodal hyperpolarization extends much farther away from the stimulus site in the direction transverse to the fiber orientation. Tear-drop shaped virtual cathodes (VCs) form along the fiber direction. This peculiar pattern of hyperpolarization is referred to as "dogbone". It should be noted that the setups used in A and B were identical except that in (A) the transverse interstitial conductivity, $g_{et}$, was reduced so that the ratio of longitudinal to transverse conductivity was the same in the intracellular and interstitial spaces. (C) The transmembrane voltage pattern induced in rabbit ventricles by application of a 1 V battery to the right ventricular free wall with the anode and cathode (CA) separated by 1 cm is shown. Battery terminals are indicated by the "+" and "−" signs. Virtual electrodes, both VCs and virtual anodes (VAs) are seen as areas of depolarization and hyperpolarization adjacent to the anode and cathode (CA) respectively. The model contained 862,000 extracellular nodes and 547,000 myocardial nodes with the membrane ionic current modeled by a modified Beeler–Reuter representation (Skouibine et al., 1999). The time instant shown is 50 ms after the application of the battery, by which time a depolarization has emanated from the site and propagated across the entire myocardium.