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On stochastic Gilpin–Ayala population model with Markovian switching

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ABSTRACT

In this paper, we analyze a stochastic Gilpin–Ayala population model with Markovian switching and white noise. The Gilpin–Ayala parameter is also allowed to switch. We establish the global stability of the trivial equilibrium state of the model. Verifiable sufficient conditions which guarantee the extinction and persistence are provided. Furthermore, we show the existence of a stationary distribution. The analytical results are illustrated by computer simulations.

1. Introduction

Ecology is the area of biology that studies the distribution and abundance of different types of organism and how these properties are affected by interactions between the organisms and their abiotic and biotic environment. The environment of an organism includes both physical properties, which can be described as the sum of local abiotic factors such as insolation, climate, and geology, as well as the other organisms that share its habitat. Population ecology is a sub-field of ecology that deals with the dynamics of species populations and how these populations interact with the environment. It is concerned with the study of how the population sizes of species living together in groups change over time and space (Begon et al., 2006). Many types of mathematical models have been proposed in the literature to provide an abstract of some significant aspect of true ecological situation. The books by Golpalsamy (1992) and Kuang (1993) are good references in this subject. The well-known and mathematically simple model used to describe the temporal evolution of a single species population in a constant environment is the logistic model described by the ordinary differential equation:

$$dx(t) = x(t)(r - kx(t))dt, \tag{1}$$

where $x(t)$ stands for the population size at time t, $r>0$ represents the growth rate of the species while $k>0$ represents the self-inhibition rate. However, the rate of change of the population size in the logistic model (1) is a linear, namely $r - kx(t)$. This way, many important factors are neglected. Therefore, many modifications of model (1) have been suggested in order to obtain more realistic solutions (Golpalsamy, 1992; Gilpin and Ayala, 1973; May, 1973). Gilpin and Ayala (1973) claimed that a little more complicated model was needed and proposed their following model:

$$dx(t) = x(t)(r - kx^{\theta}(t))dt, \tag{2}$$

where θ is a positive parameter to modify the classical logistic model.

In the real world, population systems are always influenced by stochastic perturbations. Rand and Wilson (1991) partitioned these stochastic effects into three classes: demographic, environmental and observational errors. The latter are easier to handle as they are not involved in the dynamics. The demographic stochasticity originates from the continuous approximation in a system of differential equations, instead of being described by discreet, integer-valued process (Hethcote, 1998; May, 1973). The effects of demographic stochasticity can be neglected when the population sizes are large enough. However, the effects of environmental noises remain the same for all population sizes. May (1973) has revealed the fact that because of the environmental noise, the birth rate, carrying capacity and other parameters involved in the system exhibit random fluctuation to a greater or lesser extent. Thus, the standard

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technique of parameter perturbation has become more and more popular for building stochastic population models. It is well known that a given ordinary differential equation and its corresponding stochastic equation with perturbed parameters may have significant differences. The pioneering work was due to Khasminskii (1980), who established an unstable system by using two white noise sources, and his work opened a new chapter in the study of stochastic stabilisation. Several years later, Mao and his co-workers showed in Mao et al. (2002) another important fact, namely that the environmental noise can suppress explosions in a finite time in population dynamics and this paper had an important impact on the study of stochastic population systems. In recent years, stochastic versions Gilpin–Ayala model have been studied by many authors. We mention some of the works (Jiang et al., 2008; Jiang and Shi, 2005; Jovanovic and Vasilova, 2013; Lian and Hu, 2006, 2008; Liu and Wang, 2012; Li, 2013; Vasilova and Jovanovic, 2011; Vasilova, 2013) and the references therein. Jiang et al. (2008) and Jiang and Shi (2005) considered the following non-autonomous randomized model based on (2):

$$dx(t) = x(t)(r(t) - k(t)x^{\theta}(t))dt + \alpha(t)x(t)dB(t), \tag{3}$$

where $B(t)$ is a one-dimensional Brownian motion and $\alpha^2(t)$ represents the intensity of the white noise. The authors have established the explicit global positive solution of Eq. (3). Moreover, they showed that $\mathbb{E}(1/x(t))$ has a unique positive periodic solution. They also discussed the stochastic permanence and the global attractivity of the solutions.

In fact, there are several types of environmental noise. Besides the white noise there is also the telegraph noise (Hu and Wang, 2011; Luo and Mao, 2007, 2009; Li et al., 2009; Zhu and Yin, 2009). The latter can be demonstrated as a switching between two or more regimes of environment, which differ by factors such as the availability of food and climatic characteristics. Frequently, the switching among different environments is memoryless and the waiting time for the next switch is exponentially distributed. The regime-switching can hence be modeled by a continuous-time Markov chain $(\gamma(t))_{t\geq 0}$ taking values in a finite state space $\mathcal{S} = \{1, 2, \ldots, m\}$ and having the generator $\Phi = (\phi_{uv})_{1\leq u,v\leq m}$. Recently, stochastic modeling with Markovian switching has received a great deal of attention. For instance, Takeuchi et al. (2006) investigated the evolution of a system composed of two predator–prey deterministic systems described by Lotka–Volterra equations in random environment. Du et al. (2004) studied the trajectory behavior of Lotka–Volterra competition bistable systems and systems with telegraph noise. Some asymptotic properties of the population dynamics model under regime switching have been established in the literature (Luo and Mao, 2007, 2009; Li et al., 2009). Especially, the results of Luo and Mao (2007, 2009) which showed that the positive solution of the associated stochastic differential equation does not explode in finite time with probability 1. Moreover, they demonstrated that the solution is stochastically ultimately bounded and the time average of the second moment of the solution is also bounded. Very recently, Settati and Lahrouz (2014) have investigated the positive recurrence of an n-species model of facultative mutualism under regime switching system. Li et al. (2009) discussed the stochastic permanence and extinction of a Lotka–Volterra system under regime switching, and they gave an estimation of the limit of the average in time of the sample path. Meng and Wang (2011, 2012) have studied the following switching diffusion Gilpin–Ayala population model

$$\begin{aligned} dx(t) &= (x(t)(r(\gamma(t)) - k(\gamma(t))x^{\theta_1}(t)))dt + \alpha_1(\gamma(t))x(t)dB(t) \\ &\quad + \alpha_2(\gamma(t))x^{1+\theta_2}(t)dB(t). \end{aligned} \tag{4}$$

The authors showed that Eq. (4) has a unique global positive solution for any given positive initial value. Moreover, they proved under the standing hypothesis that $(\gamma(t))_{t\geq 0}$ is ergodic with a unique stationary distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$, that the asymptotic behavior of (4) is determined by the quantity $\bar{b} = \sum_{j=1}^{m}\pi_j(r(j) - (\alpha_1^2(j)/2))$. More precisely, the authors established the following assertions:

(A1) If $\bar{b} < 0$, then the population $x(t)$ represented by (4) goes to extinction almost surely. That is,

$$\lim_{t\to\infty} x(t) = 0 \, a.s.$$

(A2) If $\bar{b} = 0$, then the population is nonpersistent in the mean. That is,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t x(t)dt = 0 \, a.s.$$

(A3) If $\bar{b} > 0$, then the population is weakly persistent. That is

$$\limsup_{t\to\infty} x(t) > 0 \, a.s.$$

(A4) If $\bar{b} > 0$, $\phi_{uv} > 0$ for all $u \neq v$, $0 < \theta_2 \leq 1$ and $0 < \theta_1 \leq 1 + \theta_2$. Then, the population is stochastically permanent. That is, for any $\epsilon \in (0, 1)$, there exists a pair of constants $\beta_1 > 0$, $\beta_2 > 0$ such that

$$\liminf_{t\to\infty} \mathbb{P}\{x(t) \geq \beta_1\} \geq 1 - \epsilon, \quad \liminf_{t\to\infty} \mathbb{P}\{x(t) \leq \beta_2\} \geq 1 - \epsilon.$$

Motivated by the above discussions, in this paper, we study the hybrid system, or system with Markovian switching of the form

$$dx(t) = (x(t)(r(\gamma(t)) - k(\gamma(t))x^{\theta(\gamma(t))}(t)))dt + \alpha(\gamma(t))x(t)dB(t). \tag{5}$$

The main concern of this paper is: if the overall system (5) consists of stable subsystems and unstable subsystems, can the overall system be stable?

The remainder of the paper is structured into four more sections as follows. In Section 2, we establish that the equilibrium state $x = 0$ of system (5) is globally asymptotically stable in probability under condition $\bar{b} < 0$. In Section 3, we show that if θ switch randomly between two or more regimes then, the population is weakly persistent under condition $\bar{b} > 0$. we studied the lower-growth rate and the upper-growth rate of the positive solutions. In addition, we estimated the limit of the average in time of the sample paths of solutions. In Section 4, we prove that under condition $\bar{b} > 0$, the system model (5) admits a unique ergodic stationary distribution. At last, we introduce some figures to illustrate the main results.

2. Global stability

Throughout this paper, we suppose that there is a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions in which the one-dimensional Brownian motion $B(t)$ is defined. Let $(\gamma(t))_{t\geq 0}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, taking values in a finite state space $\mathcal{S} = \{1, 2, \ldots, m\}$ with the generator $\Phi = (\phi_{uv})_{1\leq u,v\leq m}$ given, for $\Delta t > 0$, by

$$\mathbb{P}(\gamma(t+\Delta t) = v|\gamma(t) = u) = \begin{cases} \phi_{uv}\Delta t + o(\Delta t), & \text{if } u \neq v, \\ 1 + \phi_{uu}\Delta t + o(\Delta t), & \text{if } u = v. \end{cases}$$