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Periodic motion of four spheres in a “kite” configuration

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ABSTRACT

Cluster settling plays an important role in the sedimentation of dilute suspensions. Studies of isolated clusters indicate that they show considerable stability. In particular, spheres initially arranged in a horizontal isosceles triangle or rhombus exhibit simple periodic motion as they descend. Moving one sphere outward along the long diagonal of a rhombus produces a kite configuration that settles with a complicated periodic motion relative to its center of mass, which exhibits a back-and-forth movement as it descends. This makes the kite configuration the second member of a class between small clusters with simple periodic motions and isolated clusters that merely remain intact for a considerable time. If the displacement along the diagonal is too great, one sphere is left behind. Higher-order terms, which have little effect on widely separated spheres in a rhombus, are important in maintaining the periodicity of the kite configuration and in slowing its breakup.

1. Introduction

Interest in the sedimentation of small clusters of identical spheres arises from two different sources. One, summarized by [1], is the increasing power of theoretical analyses to handle simple systems of several spheres. The other is the recognition of the importance of cluster settling, especially in very dilute suspensions [2–12]. In suspensions with $0.005<\varphi<0.03$ (where φ is the solids fraction), the important feature is not close pairs, but fairly close neighbors [10]. These settle as a transient, loose cluster. Some spheres break off and other spheres join [9,11,12]. As the denser regions move rapidly downward, spheres in more dilute regions may be stationary or even move upward [9]. The interest in cluster settling in suspensions led to experimental studies of isolated dense clusters [13–15]. These showed considerable stability: some clusters broke up or shed spheres as they settled, but others remained intact. These could be modeled as a liquid drop whose density and viscosity were determined by the value of φ within the boundary of the cluster [14]. There have also been some computational studies of fairly large clusters of identical spheres [16,17].

Interest in the theoretical and computational aspects of cluster settling was spurred by an experimental study of the sedimentation of spheres arranged in regular polygons [18,19]. Though the earliest work focused on the three-sphere isosceles triangle [20–22], the four-sphere cluster is the prototype of isolated clusters that circulate as they travel downward [13,14]. Tory et al. [23] showed that four spheres initially arranged in the shape of a horizontal rhombus exhibited a periodic motion. The orbits with respect to the center of mass ranged from almost elliptical for very slight deviations from a square [24] to egg-shaped for fairly small deviations. Larger deviations led to orbits with two inflection points. These orbits became very elongated for large deviations. Finally, the two closely spaced spheres broke away if the other two spheres were sufficiently far apart [23]. Unlike spheres in an isosceles triangle, whose mean velocity $\mathbf{u}_m$ changes direction as they descend, $\mathbf{u}_m=u_m\mathbf{k}$ for spheres arranged in a rhombus [23]. The variation of u_m with time is approximately sinusoidal [23].

An outward displacement of one of the spheres on the long diagonal of a rhombus produces a “kite” configuration (Fig. 1). Simulations [23,25] in which the displacement was small showed that the orbit of the displaced sphere was confined to a narrow band. Larger displacements produced a broader band and still larger displacements led to very complicated paths. Nevertheless, the cluster usually remained intact. In many respects, these results parallel those for three spheres. Those arranged in an isosceles triangle [20–22] exhibited periodic motion; asymmetric arrangements exhibited nearly periodic motion and the three-sphere cluster remained intact [21]. Snook et al. [26] found complicated periodicities in the trajectories of spheres slightly displaced from an isosceles triangle. This suggested that the kite configuration might show similar behavior.

2. Mathematical treatment

Our treatment is based on an idealization known as the steady Stokes equations, which also apply to unsteady flow when the terms related to time-dependence are negligible. The particles are identical

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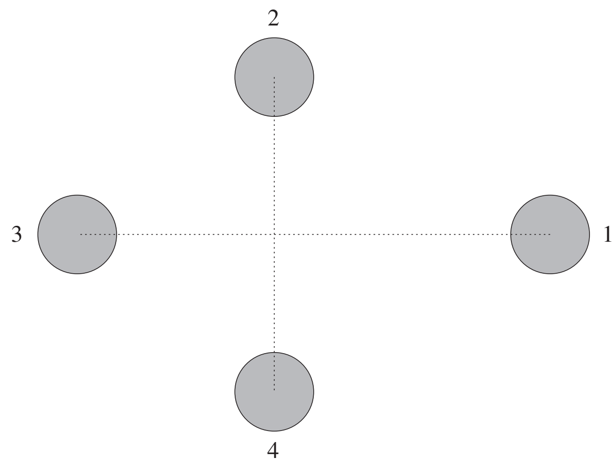

Fig. 1. Kite configuration. Initially, all spheres are in the same horizontal plane.

spheres and the fluid is Newtonian and incompressible. The theoretical solution can be approximated experimentally by ensuring that the spheres are large enough that Brownian motion is negligible, but small enough that inertial effects are also negligible. In experimental work, this balance is often achieved by using a fluid that is much more viscous than water [9,12,32]. Boundaries must be distant and the only force is that of gravity.

In the treatment that follows, distance is made dimensionless by dividing by the sphere radius a, velocities by dividing by the Stokes velocity u_0, and time by multiplying by u_0/a.

2.1. Kite configuration

Fig. 1 shows a kite configuration with spheres at dimensionless positions $(x_1,0,z_1)$, (x_2,y_2,z_2), $(x_3,0,z_3)$, and (x_4,y_4,z_4) where $x_1>-x_3$ initially. Though some of the symmetry of the rhombus is lost, some symmetries remain, viz., $x_4=x_2$, $y_4=-y_2$, $z_4=z_2$. Clearly, these symmetries will be preserved throughout the descent.

Coupled linear equations relate the translational and rotational velocities of particles to the forces and torques that they exert on the fluid [27,28]. The torque on a freely rotating sphere is zero, so we need to consider only the forces. With terms to $O(r^{-7})$, the dimensionless velocity of each sphere is given by

$$-\mathbf{u}_i = (\mathbf{A}_{i1} + \mathbf{A}_{i2} + \mathbf{A}_{i3} + \mathbf{A}_{i4})\cdot\mathbf{k}, \tag{1}$$

where $\mathbf{A}_{ij}$ is the dyadic (second-rank tensor) given by Eq. (7) of Kamel and Tory [27]. This equation is a dimensionless version of results from Table II and Eqs. (6.19)–(6.25) of Mazur and van Saarloos [28]. Positions and velocities are positive upward. Ladd's method [29] is useful for spheres that are close together [22,26]. For two simple cases, Ladd [29] has shown that it gives very nearly the same results as Eq. (1) with terms to $O(r^{-7})$. For widely separated spheres in simple periodic motion, terms of $O(r^{-1})$ are often sufficient [23]. However, the much greater accuracy of Ladd's method serves as a check and also covers many more cases.

2.2. Equations for the velocities of widely separated spheres

In our earlier work on the sedimenting rhombus [23], a study of widely separated spheres proved useful in understanding their orbits. For simplicity, we will use only the first two terms of A_{ij} for spheres that are far apart, reserving Ladd's method for the general case. Thus

$$\mathbf{A}_{ij} \approx \delta_{ij}\mathbf{I} + 3\left(\mathbf{I} + \hat{\mathbf{r}}_{ij}\hat{\mathbf{r}}_{ij}\right)/\left(4r_{ij}\right) \tag{2}$$

where $\hat{\mathbf{r}}_{ij}$ is a unit vector and r_{ij} is the dimensionless distance from sphere i to sphere j. From Eqs. (1) and (2), a straightforward calculation using the symmetries noted above yields

$$\begin{aligned}-\mathbf{u}_1 = {} & \left[1+3/(2r_{12}) + 3/(4r_{13}) + 3z_{12}^2/\left(2r_{12}^3\right) + 3z_{13}^2/\left(4r_{13}^3\right)\right]\mathbf{k} \\ & + \left[3x_{12}z_{12}/\left(2r_{12}^3\right) + 3x_{13}z_{13}/\left(4r_{13}^3\right)\right]\mathbf{i},\end{aligned} \tag{3}$$

$$\begin{aligned}-\mathbf{u}_2 = {} & \left[1 + 3/(4r_{12}) + 3/(4r_{23}) + 3/(4r_{24}) + 3z_{12}^2/\left(4r_{12}^3\right)\right. \\ & \left. + 3z_{23}^2/\left(4r_{23}^3\right)\right]\mathbf{k} + \left[3x_{12}z_{12}/\left(4r_{12}^3\right) + 3x_{23}z_{23}/\left(4r_{23}^3\right)\right]\mathbf{i} \\ & + \left[3y_{12}z_{12}/\left(4r_{12}^3\right) + 3y_{23}z_{23}/\left(4r_{23}^3\right)\right]\mathbf{j},\end{aligned} \tag{4}$$

$$\begin{aligned}-\mathbf{u}_3 = {} & \left[1 + 3/(4r_{13}) + 3/(2r_{23}) + 3z_{13}^2/\left(4r_{13}^3\right) + 3z_{23}^2/\left(2r_{23}^3\right)\right]\mathbf{k} \\ & + \left[3x_{13}z_{13}/\left(4r_{13}^3\right) + 3x_{23}z_{23}/\left(2r_{23}^3\right)\right]\mathbf{i},\end{aligned} \tag{5}$$

$$\begin{aligned}-\mathbf{u}_4 = {} & \left[1 + 3/(4r_{12}) + 3/(4r_{23}) + 3/(4r_{24}) + 3z_{12}^2/\left(4r_{12}^3\right)\right. \\ & \left. + 3z_{23}^2/\left(4r_{23}^3\right)\right]\mathbf{k} + \left[3x_{12}z_{12}/\left(4r_{12}^3\right) + 3x_{23}z_{23}/\left(4r_{23}^3\right)\right]\mathbf{i} \\ & - \left[3y_{12}z_{12}/\left(4r_{12}^3\right) + 3y_{23}z_{23}/\left(4r_{23}^3\right)\right]\mathbf{j},\end{aligned} \tag{6}$$

where

$$x_{ij} = x_j - x_i, \quad y_{ij} = y_j - y_i, \quad z_{ij} = z_j - z_i, \tag{7}$$

$$r_{ij} = \left(x_{ij}^2 + y_{ij}^2 + z_{ij}^2\right)^{1/2}, \tag{8}$$

and $\mathbf{u}_i = \mathbf{i}dx_i/dt + \mathbf{j}dy_i/dt + \mathbf{k}dz_i/dt$. Owing to the symmetry of y_2 and y_4, $\mathbf{x}_4 = (x_4, y_4, z_4)$ can be calculated from $\mathbf{x}_2$. Thus, there are only seven scalar differential equations to solve. These, together with the symmetry condition, yield the trajectories of the four spheres.

2.3. Equations for the velocities relative to the center of mass

The mean velocity of the four spheres is

$$\begin{aligned}-\mathbf{u}_m = {} & \left[1+3/(4r_{12}) + 3/(8r_{13}) + 3/(8r_{24}) + 3/(4r_{23}) + 3z_{12}^2\right. \\ & \left. \div\left(4r_{12}^3\right) + 3z_{13}^2/\left(8r_{13}^3\right) + 3z_{23}^2/\left(4r_{23}^3\right)\right]\mathbf{k} + \left[3x_{12}z_{12}/\left(4r_{12}^3\right)\right. \\ & \left. + 3x_{13}z_{13}/\left(8r_{13}^3\right) + 3x_{23}z_{23}/\left(4r_{23}^3\right)\right]\mathbf{i}.\end{aligned} \tag{9}$$

For the rhombus, $z_{13}=0$, $r_{12}=r_{23}$, $x_{12}=x_{23}$, and $z_{12}+z_{23}=0$, which eliminates the term in $\mathbf{i}$. Thus, the kite configuration (unlike the rhombus) has a mean velocity with a horizontal component. Eq. (9) shows that $y_m(t)=0$ for both configurations, where y_m is the y-component of the center of mass. We can easily derive the velocities relative to the center of mass

$$\begin{aligned}-\mathbf{u}_1 + \mathbf{u}_m = {} & (3/4)\left[1/r_{12} + 1/(2r_{13}) + z_{12}^2/r_{12}^3 + z_{13}^2/\left(2r_{13}^3\right)\right. \\ & \left. -1/(2r_{24}) - 1/r_{23} - z_{23}^2/r_{23}^3\right]\mathbf{k} + (3/4)\left[x_{12}z_{12}/r_{12}^3\right. \\ & \left. + x_{13}z_{13}/\left(2r_{13}^3\right) - x_{23}z_{23}/r_{23}^3\right]\mathbf{i}\end{aligned} \tag{10}$$

$$\begin{aligned}-\mathbf{u}_2 + \mathbf{u}_m = {} & (3/8)\left[1/r_{24} - 1/r_{13} - z_{13}^2/r_{13}^3\right]\mathbf{k} - (3/8)\left[x_{13}z_{13}/r_{13}^3\right]\mathbf{i} \\ & + (3/4)\left[y_{12}z_{12}/r_{12}^3 + y_{23}z_{23}/r_{23}^3\right]\mathbf{j}\end{aligned} \tag{11}$$