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# Interactive software for calculating the principal stresses of compacted cohesive powders with the Warren-Spring equation

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ABSTRACT

The Unconfined Yield Stress ($\sigma_c$) and Major Consolidation Stress ($\sigma_1$) of a cohesive powder's compact are found by constructing two Mohr semicircles that are tangential to the Yield Loci Curve (YLC); the first passing through the origin (0,0) and the second at the consolidation conditions ($\sigma_0$,$\tau_0$). When the YLC can be described by the Warren-Spring equation $(\tau/C)^n=(\sigma+T)/T$ or an alternative algebraic expression, this translates into finding the solution of two pairs of simultaneous equations that set the conditions for the tangential YLC and corresponding Mohr semicircles to have the same value and slope at their respective contact points. Once the Mohr semicircle's equation that corresponds to the consolidation conditions has been found, the Effective Angle of Internal Friction ($\delta$) is calculated in a similar manner. The numerical calculation procedure has been automated in a freely downloadable program posted on the web as a Wolfram Project Demonstration. It allows the user to choose and adjust the values of $C$, $T$, $n$ and $\sigma_0$, and the plot's scales, by moving sliders on the computer screen. The program calculates and displays the corresponding values of $\sigma_c$, $\sigma_1$ and $\delta$, and plots the YLC, two Mohr semicircles and the line that defines $\delta$. Since a linear YLC is just a special case of the model where $n=1$, the program can be used with input parameters originally obtained by linear regression. But although the program can render reasonable estimates of the principal stresses $\sigma_c$, $\sigma_1$ and $\delta$ in this case too, the physical meaning of $C$, and especially $T$, is unclear when calculated by extrapolation instead of being determined experimentally.



## 1. Introduction

The flowability of cohesive powders has been primarily determined through shear analysis. The tests are performed on powder specimens consolidated and sheared under various compressive stresses, and are sometimes complemented by tensile strength measurements. The test results of specimens consolidated under the same normal (compressive) stress are in the form of a 'Yield Loci Curve', shown schematically in Fig. 1. From this Yield Loci Curve, one can obtain the compact's 'major consolidation stress', $\sigma_1$, and its 'Unconfined Yield Stress', $\sigma_c$, which is also shown in the figure. The relationship between $\sigma_c$ and $\sigma_1$, obtained by testing the powder under several consolidation stresses is known as the 'Flow Function'. It is frequently used in bin and hopper design [1–7]. In contrast with free flowing powders, which have a 'low' Flow Function, a cohesive powder has a 'high' Flow Function, indicating that it develops considerable strength when consolidated under a normal stress. The Flow Function, together with the powder's Effective Internal Angles of Friction ($\delta$), and the wall's, determines bin geometries and aperture sizes that are appropriate for gravitational flow.

The experimental Yield Loci Curves of many cohesive powders can be described by the Warren-Spring equation [8]:

$$\left(\frac{\tau}{C}\right)^n = \frac{\sigma + T}{T} \tag{1}$$

where $\tau$ is the observed yield stress in shear, $C$ is the compact's 'cohesion', $\sigma$ the normal consolidation stress and $T$ the compact's tensile strength, all in stress units, and $n$ a dimensionless 'curvature index' ($1 \le n \le 2$). In some cases, experimental Yield Loci Curve, especially when far enough from the shear stress axis (see Fig. 2), can be described by a straight line, known as the Coulomb Equation, i.e.,

$$\tau = C + b\sigma \tag{2}$$

where $C$ is the straight line's intercept with the $\tau$ axis and $b$ its slope.

Formally, Eq. (2) could be considered a special case of Eq. (1) where $C$ is the Cohesion and $b=C/T$. In reality, however, such a relationship between $C$ and $T$ is probably very rare in cohesive powders.

There have also been attempts to improve on the Warren-Spring equation [9], and to link the cohesion and tensile strength through a power-law relationship [10], which could be incorporated in the Yield Loci Curve's equation. But by and large, Eq. (1) has been the most

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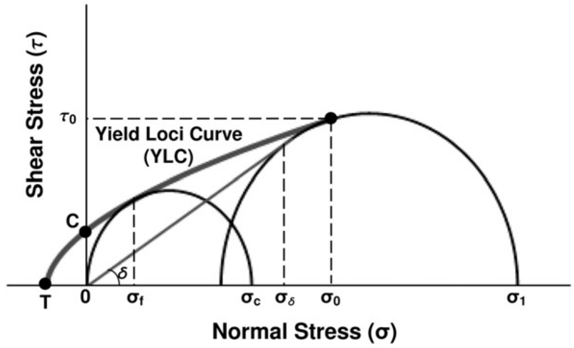


**Fig. 1.** A Yield Loci Curve generated with the Warren-Spring equation, the two Mohr semicircles and corresponding Unconfined Yield Stress, $\sigma_c$, and major consolidation stress, $\sigma_1$. *C* and *T* are the compact's 'cohesion' and tensile strength, respectively. The effective angle of friction, $\delta$, is also shown.

commonly used model to describe experimental Yield Loci Curve mathematically, especially when the tensile stress (*T*) has been independently determined.

Finding $\sigma_c$ and $\sigma_1$ can be done graphically or by calculation (7). The latter can be done in different ways. For this work, we have chosen the one based on that at the point of contact between the Yield Loci Curve and tangential Mohr semicircle, the two have the same numerical value and also the same local slope.

## 2. Calculation of the Unconfined Yield Stress, $\sigma_c$, and Principal Consolidation Stress, $\sigma_1$, using the Warren-Spring equation

### 2.1. Calculation of $\sigma_c$

A semicircle whose center lies on the $\sigma$ axis is described by the equation:

$$\tau = \sqrt{R^2-(\sigma-M)^2} \tag{3}$$

where *R* is its radius and *M* its center's location along the σ axis.

Since by definition (see Fig. 1) $\sigma_c = 2M$ and $R = M$, Eq. (3) becomes:

$$\tau = \sqrt{(\sigma_c-\sigma)\sigma} \tag{4}$$

At the point where the Mohr semicircle is tangential to the Yield Loci Curve, the point ($\sigma_f$, $\tau_f$) in the figure, its local slope is:

$$\left.\frac{d\tau}{d\sigma}\right|_{\sigma_f\tau_f} = \frac{\sigma_c-2\sigma_f}{2\sqrt{(\sigma_c-\sigma_f)\sigma_f}} \tag{5}$$

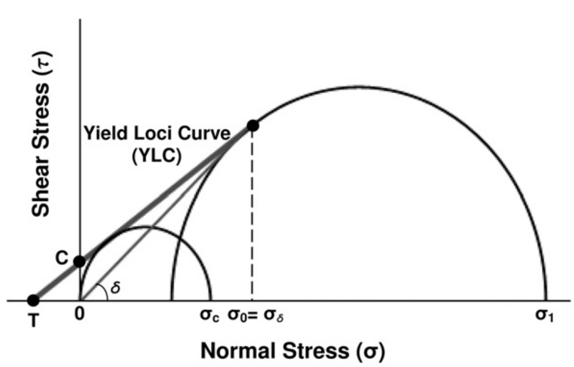


**Fig. 2.** A "linear" Yield Loci Curve frequently observed when the compact's tensile strength (*T*) and 'cohesion' (*C*) have not been measured. Notice that although data in the 'linear region' can be used to estimate the principal stresses, extrapolation of the line will most probably render incorrect values of both *T* and *C*. Also notice that in the linear case, the tangent line meets the larger Mohr semicircle at the consolidation conditions.

At this point of contact ($\sigma_f$,$\tau_f$), the local slope of the Yield Loci Curve when described by the Warren-Spring equation is:

$$\left.\frac{d\tau}{d\sigma}\right|_{\sigma_f\tau_f} = \frac{C\left(1+\frac{\sigma_f}{T}\right)^{\frac{1}{n}-1}}{nT} \tag{6}$$

Consequently, we have the two equalities:

$$\tau = C\left(1+\frac{\sigma_f}{T}\right)^{\frac{1}{n}} = \sqrt{(\sigma_c-\sigma_f)\sigma_f} \tag{7}$$

which signifies that they Yield Loci Curve and Mohr semicircle share the same point ($\sigma_f$, $\tau_f$) and

$$\frac{d\tau}{d\sigma} = \frac{C\left(1+\frac{\sigma_f}{T}\right)^{\frac{1}{n}-1}}{nT} = \frac{\sigma_v-2\sigma_f}{2\sqrt{(\sigma_c-\sigma_f)\sigma_f}} \tag{8}$$

which signifies that the two have the same slope at this point.

Eqs. (7) and (8) are two simultaneous algebraic equations that have two unknowns, namely $\sigma_c$, the 'Unconfined Yield Stress' in which we are interested, and $\sigma_f$ the location of the point where the Mohr semicircle and Yield Loci Curve are tangential.

The two equations can be easily solved numerically using the 'FindRoot' of Mathematica® (Wolfram Research, Champaign, IL), the program we used in this work, but also by equation solvers of other commercial mathematical software. Once the two equations are solved, which is done almost instantaneously, the program renders the value of $\sigma_c$ and plots the corresponding Mohr semicircle—see below. A similar procedure can be developed for YLC's that are described by alternative algebraic expressions.

### 2.2. Calculation of $\sigma_1$

The Major Principal Stress at the initial consolidation conditions, i.e., where $\sigma = \sigma_0$ and $\tau = \tau_0$ is known as the 'Major Consolidation Stress' and called $\sigma_1$—see Fig. 1. From basic geometry, the Mohr semicircle that intersects with the $\sigma$-axis at the point $\sigma_1$ has to follow the equation:

$$\tau = \sqrt{(\sigma_1-M)^2-(\sigma-M)^2} \tag{9}$$

where *M*, as before, is the semicircle center's location along the $\sigma$-axis.

The slope of this Mohr semicircle at its contact point with the Yield Stress Loci Curve ($\sigma_0$, $\tau_0$) is therefore:

$$\left.\frac{d\tau}{d\sigma}\right|_{\sigma_0\tau_0} = \frac{M-\sigma_0}{\sqrt{(\sigma_1-M)^2-(\sigma_0-M)^2}} \tag{10}$$

Here again, since the Yield Loci Curve and its tangential Mohr semicircle have the same numerical value and slope at their contact point, ($\sigma_0$ $\tau_0$) in this case, we have to solve two simultaneous equations in order to extract the principal stress. They are:

$$C\left(1+\frac{\sigma_0}{T}\right)^{\frac{1}{n}} = \sqrt{(\sigma_1-M)-(\sigma_0-M)^2} \tag{11}$$

and

$$\frac{C\left(1+\frac{\sigma_0}{T}\right)^{\frac{1}{n}-1}}{nT} = \frac{M-\sigma_0}{\sqrt{(\sigma_1-M)^2-(\sigma_0-M)^2}} \tag{12}$$

The two unknowns here are $\sigma_1$, the Major Consolidation Stress in which we are particularly interested, and *M* the Mohr semicircle's location along the $\sigma$-axis. These two simultaneous equations too can