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Reprint of "Mathematics as verbal behavior"☆,☆☆

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ABSTRACT

"Behavior which is effective only through the mediation of other persons has so many distinguishing dynamic and topographical properties that a special treatment is justified and indeed demanded" (Skinner, 1957, p. 2). Skinner's demand for a special treatment of verbal behavior can be extended *within* that field to domains such as music, poetry, drama, and the topic of this paper: mathematics. For centuries, mathematics has been of special concern to philosophers who have continually argued to the present day about what some deem its "special nature." Two interrelated principal questions have been: (1) Are the subjects of mathematical interest pre-existing in some transcendental realm and thus are "discovered" as one might discover a new planet; and (2) Why is mathematics so effective in the practices of science and engineering even though originally such mathematics was "pure" with applications neither contemplated or even desired? I argue that considering the actual practice of mathematics in its history and in the context of acquired verbal behavior one can address at least some of its apparent mysteries. To this end, I discuss some of the structural and functional features of mathematics including verbal operants, rule-and contingency-modulated behavior, relational frames, the shaping of abstraction, and the development of intuition. How is it possible to understand Nature by properly talking about it? Essentially, it is because nature taught us how to talk.

1. Introduction

"It is still an unending source of surprise for me to see how a few scribbles on a blackboard or on a sheet of paper could change the course of human affairs". Stanislaw Ulam.

In this paper I am revisiting topics I have wrestled with off and on over decades, but an opportunity to speak at the Society for the Quantitative Analysis of Behavior (SQAB) in May of 2014 allowed me to reconsider some of my old papers (Marr, 1986, 1995, 2003a, 2003b) as well as new sources and provide—largely from the perspective of a behavior analyst—a more up-to-date picture of the immensely complex and surprisingly contentious practice of mathematics. The scope—breadth as well as depth—of the topic is enormous, encompassing some of the most vexing problems in our struggle to understand Nature. How has that knowledge come about and evolved, and how it is possible to grasp and predict the patterns of nature by talking about them in the special ways we call mathematics? Even more puzzling, initially many of these special ways seem merely the product of constrained verbal play, having no apparent relations whatever to our experiences of nature. To begin to address such issues, as any behavior analyst would assert, one needs to look at the history of mathematics as well as its actual practice. With respect to practice, we might look at aspects of its acquisition and controlling variables in the light of current understanding of the dynamics of verbal behavior.

1.1. What is Mathematics?

One should not be surprised that there is no consensus on a definition of such a "motley" enterprise (cf. Wittgenstein, 1978 #46, p.176). Hacking (2014) lists 17 different definitions gleaned for various dictionaries. Davis and Hersh (1981) provide a provisionary description as the *science of quantity and space including the associated symbolism*. Another definition is the *science of patterns* (Devlin, 1994). However vague, I find this latter definition appealing because the origin of all sciences is found in pattern recognition and generation, and mathematics is the most powerful method we have for characterizing those patterns we find in nature as well as

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generating new ones. Strongly interacting with our mathematical descriptions of nature are those patterns engendered within the exercise of mathematics itself.

Traditionally, mathematics is divided into the domains of algebra, geometry and analysis (e.g., Gowers, 2008), but this is far too simplistic, if only because of the considerable overlap of these areas in modern mathematics. Moreover, other fields not so easily classified have emerged in the last century or so. One estimate of the number of distinguishable areas in current mathematical practice exceeds 3000, an estimate Davis and Hersh made in 1981! Clearly, no mathematician could have a practicing knowledge of anything but a very tiny percentage of such a huge edifice.

1.2. But what's it all about?

In raising this question, I am attempting to address what philosophers might describe as the *Nature* of mathematical knowledge. I raise this question because in briefly addressing it, and adopting a point of view, I think we may be led to consider the practice of mathematics in the light of a *behavioral* science—or, at the least, free ourselves from the magic and mysticism that are so often attributed to mathematics. One cannot begin to approach the analysis of mathematics as verbal behavior without a close study of how mathematicians actually work, and especially, how they have acquired their skills. Formal philosophical treatments of both science and mathematics were especially popular in much of the last century, but both efforts foundered on attempts to achieve foundations. To have any grasp of these seemingly lofty enterprises, one has to bring them down to earth through explorations of the actual practices of scientists and mathematicians—in other words, provide a *behavior* analysis. In the light of actual practices, foundations evaporate.

The philosophy of mathematics has had a long and checkered history that burned brightly with controversy in the last century, but while a good bit of that heat has dissipated, some of the issues still smolder (e.g., Hacking, 2014). Much of these concerns involve two interrelated questions: (1) what is the status of mathematical truth; and (2) what is the nature of mathematical objects? As Gowers (2006) expresses it, "...what gives mathematical statements their aura of infallibility, and what on earth are these statements about?" (p. 182). In addressing such questions, many different "schools" developed: Platonism, logicism, formalism, nominalism, intuitionism, and empiricism to name the major ones.

In the interest of brevity as well as relevance, I will focus here only on two opposing views—*Platonism* (sometimes, curiously, called realism) and *empiricism*. There are varieties of each, but I will ignore much of that detail (Hersh, 1997; Hacking, 2014).

1.3. Platonism

The Platonist or realist school believes in the reality of mathematical entities as eternal, objective facts independent of human existence; in other words, something like Plato's ideal forms. While this position was only named in the 20th century (Hacking, 2014), it is of considerable age. The 19th century mathematician Hermite expressed the view as follows:

I believe that the numbers and functions of analysis...exist outside of us with the same character of necessity as the objects of objective reality; and we find or discover them and study them as do the physicists, chemists and zoologists (quoted by Kline, 1980; p. 322).

Thus, mathematics is placed alongside other natural sciences, and the mathematician, by implication, is a *discoverer*, not a creator. As G. H. Hardy (1967) noted:

I believe that mathematical reality lies outside of us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations' are simply our notes of our observations (pp. 123–124). Hersh (1997) remarks that this position is analogous to that of a botanist who can discover and classify plants, but cannot create new species.

The perspective of Platonism is compelling by any account. First, it is a view that most of us naively adopt as a result of our training in elementary mathematics and it is probably implicit in the activities of most practicing mathematicians. As Davis and Hersh (1981) observe: "Platonism was and is believed by (nearly) all mathematicians. But like an underground religion, it is observed in private and rarely mentioned in public" (p. 339). We are so imbued with the reification of the integers, that it does not seem so odd to consider them as pre-existing entities, but then what about π, or $\sqrt{-1}$, or infinite-dimensional vector spaces, or Cantor's transfinite numbers, or a function discontinuous at every point? The list of such mathematical "objects" is beyond counting and ever growing! What can it mean to say that there exist mathematical entities? Clearly, one is not talking about objects in the environment (other, than perhaps, marks on paper). For the Platonist, "These objects are, of course, not physical or material. They exist outside the space and time of physical existence. They are immutable—they are not created and they will not change or disappear" (Davis and Hersh, 1981; p. 318).

If we put such a view alongside Cartesian dualism, we now apparently have a *trinity*—mind, matter, and immutable, eternal forms outside of mind or matter. Actually, mind and matter would seem completely subservient to those eternal forms, for these are, in the words of one mathematician, "..., the essence of what drives the universe" (cited in Hersh, 1997, p. 10) thus, mathematics is the ultimate "intelligent design"! We may well ask as does Hersh (1997, p. 11): "Why do mathematicians believe something so unscientific, so far-fetched as an independent immaterial timeless world of mathematical truth?"

From the perspective of a competent practicing mathematician, these beliefs are perhaps not so difficult to understand as they seem inherent in the experience of doing mathematics. Certainly, one can have the strong *feeling of discovery* of one or more of these eternal entities putatively driven by what Wittgenstein (1978) called "the hardness of the logical must." What is not being brought to bear on these beliefs is *history*—both the history of mathematics over millennia, and, at least as important, the individual's history in acquiring and practicing their mathematical craft. I will return to these elements shortly. But now for an alternative perspective of the content and practice of mathematics—empiricism.

1.4. Empiricism

As opposed to believing that the universe is somehow built from pre-existing mathematical forms and relations which we discover, one can turn this perspective on its head by proposing that nature evinces patterns in its phenomena and that we *invent* the means to describe those patterns. Why does mathematics work so well? This issue was addressed in a classic essay by Wigner (1984) entitled "The unreasonable effectiveness of mathematics in the natural sciences". So, what is the problem here? Let me start with an anecdote from Wigner. A statistician was explaining to a non-mathematical friend the Gaussian, or normal distribution describing characteristics of a population:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp \frac{-(x-\mu)^2}{2\sigma^2}$$

The friend was skeptical and asked "How can you know that?" "And what is this symbol here?" "Oh," said the statistician, this is