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Original Research Article

Analytical solution for beams with multipoint boundary conditions on two-parameter elastic foundations

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ABSTRACT

An efficient analytical method is presented for the closed form solution of continuous beams on two-parameter elastic foundations. The general form of the governing equation is reduced to a system of first-order differential equations with constant coefficients. The system is then solved using Jordan form decomposition for the coefficient matrix and construction of the fundamental solution. Common types of boundary conditions (pinned and roller support, hinge connection, fixed and free end) can be applied to an arbitrary point on the beam. The method has a completely computer-oriented algorithm, computational stability, and optimal conditionality of the resultant system and is a powerful alternative to the analytical solution of beams with multipoint boundary conditions on one- or two-parameter elastic foundations. Examples with different types of loading, boundary conditions, and foundation are presented to verify the method.

1. Introduction

Recent developments in computer science and mathematics and the need for accurate solutions to problems have resulted in the development of analytical [1] and semi-analytical or discrete-continual methods [2]. For practical problems, it is often more suitable and easier to use an analytical solution than to employ an expensive finite element method (FEM)-based software. Analytical solutions employ a mathematical expression that yields the values of the unknown quantities at any location on a body (the total structure or physical system of interest) and are valid for an infinite number of locations. This property considerably reduces the computational complexity of the problem, an issue that requires special consideration in numerical methods [3].

Akimov and Sidorov [4] proposed an analytical solution to multipoint boundary problems for systems of ordinary differential equations with piecewise constant coefficients. Their method can be applied to continuous beams, which is a leading engineering problem. Beams in different types of bridges, continuous beams of multi-span girders [5], long strip foundations of buildings, and railroad and retaining walls [6] are included in this type of problem. The most important application

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of the proposed method is an analytical solution to continuous beams on elastic foundations. The method can also be applied to beams with structural or foundation discontinuities (changes in the elastic property or cross-section of the beam or stiffness of the soil) for beams without foundations by changing the coefficient matrix at the relevant interval.

Section 3 presents the formulation of the problem. Section 4 describes the analytical solution to the resultant multipoint boundary problem in detail. Various types of boundary conditions are introduced in Section 5. Section 6 uses numerical examples to demonstrate the efficiency, accuracy, and validity of the method. Section 7 presents the concluding remarks.

2. Beam on elastic foundation

A computational model of a beam on an elastic foundation is often used to describe different engineering problems [7]. Much literature has been devoted to evaluation of the behavior of soil-structure interaction (beam, in this case) [8–22]. Some [8–13] present the soil as an idealized Winkler model [23] in which disconnected soil springs are used to represent the compressive resistance of soil. These Winkler springs are characterized by the spring constant k_s, which is often related to the soil subgrade modulus. Lin and Adams [8] and Kaschiev and Mikhajlov [9] investigated the problem of beams on tensionless Winkler foundations. DasGupta [10], Banan et al. [11] and Aköz and Kadioğlu [12] researched finite element approaches for the solution of beams on elastic Winkler foundations.

In the analytical field, Borák and Marcián [13] proposed a modified Betti's theorem for an analytical solution to beams on elastic foundations. The results were acceptable and the ground behavior could be more realistically simulated than the Winkler model. Mechanical resistance in soil arises from both compressive and shear strains; thus, it is more realistic to consider shear interactions between the soil springs, which distributes ground displacement and stress beyond the loaded region. This leads to a simplified-continuum model [14] in which shear interaction is mathematically taken into account by introducing a shear parameter t_s. This parameter represents the shear force at any vertical section of the foundation. Considering the general nature of the governing differential equation describing beam deflection on a two-parameter (k_s and t_s) foundation, t_s can also be interpreted as the tensile force in the membrane connecting the soil springs. In this way, the tensile resistance generated in the ground from the placement of geosynthetics can also be taken into account [15].

Zhaohua et al. [16], Karamanlidis et al. [17], Razaqpur and Shah [18], and Morfidis et al. [19] researched a finite element solution to beams resting on two-parameter elastic foundations. Beams on three-parameter elastic foundation were studied by Avramidis and Morfidis [20] and Morfidis [21]. Dinev [22] proposed an analytical solution to a beam on an elastic foundation using singularity functions and considering two parameters for the soil model. This is applicable only for the solution to problems without special external boundary conditions; however, many practical problems have different external boundary conditions that must be considered, such as continuous strip foundations resting on piles, railroad foundations, and fixed-end foundations. The present paper proposes an exact analytical solution to beams with multiple external boundary conditions. In this method, common types of boundary conditions such as pinned and roller supports, hinge connections, and fixed and free ends at arbitrary points along the beam also allows consideration of an elastic foundation with one or two soil parameters.

3. Formulation of problem

The derivation of a field equation is based on variation in the total potential energy function and employs the following assumptions [22]:

- The beam and the soil materials are linearly elastic, homogeneous and isotropic;
- The displacements are small compared to the beam thickness;
- The axial strains are small compared to unity;
- The transversal normal strains and the shear stresses are negligibly small;
- The cross-sections are plane and perpendicular to the longitudinal axis before and after deformation (Bernoulli hypothesis).

$$\frac{\partial^4 w}{\partial x^4} - \alpha \frac{\partial^2 w}{\partial x^2} + \beta w = F(x) \tag{1}$$

$$\alpha = \frac{2t_s}{EI}, \quad \beta = \frac{k_s}{EI}, \quad F(x) = \frac{P}{EI}\delta(x - x_0) \tag{2}$$

Here, w is vertical displacement of the beam, α and β are soil parameters, and F(x) is the load applied to the structure and is represented by the delta function in distribution theory [24]. A typical beam on an elastic foundation with different types of loading and boundary conditions is shown in Fig. 1. Consider the following relations:

$$\begin{cases} w'(x) = \theta(x); \\ \theta'(x) = \dfrac{M(x)}{EI}; \\ M'(x) = Q(x); \\ Q'(x) = F(x) + \dfrac{\alpha M(x)}{EI} - \beta w(x) \end{cases} \tag{3}$$

The assumed sign convention for moments, shear force, deflection, and cross-section rotation for Eq. (3) are presented in Fig. 2. The governing differential equation of the problem (Eq. (1)) can now be transformed into a system of four differential equations of first order in matrix form as:

$$\bar{y}'(x) = \bar{f}(x) + A\bar{y}(x) \tag{4}$$

$$\bar{y}(x) = [y_1(x) \quad y_2(x) \quad y_3(x) \quad y_4(x)]^T \tag{5}$$

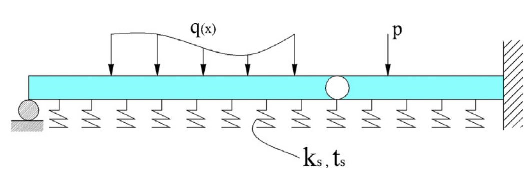

Fig. 1 – Beam on elastic foundation with different types of loading and boundary conditions.